THE HALF-INFINITE XXZ CHAIN IN ONSAGER'S APPROACH

P. BASEILHAC AND S. BELLIARD

Abstract. The half-infinite XXZ open spin chain with general integrable boundary conditions is considered within the recently developed 'Onsager's approach'. Inspired by the finite size case, the transfer matrix is simply expressed in terms of the elements of a new type of current algebra recently introduced. In the massive regime $-1 < q < 0$, level one infinite dimensional representation ($q-$vertex operators) of the new current algebra are used in order to diagonalize the transfer matrix. For diagonal boundary conditions, known results of Jimbo *et al.* are recovered. The solution for upper (or lower) non-diagonal boundary conditions is also described. In both cases, vacuum and excited states are formulated within the representation theory of the current algebra. Finally, for q generic the long standing question of the hidden non-Abelian symmetry of the Hamiltonian is solved: it is either associated with the $q-$Onsager algebra (generic non-diagonal case) or the augmented $q-$Onsager algebra (generic diagonal case).

MSC: 81R50; 81R10; 81U15.
Keywords: XXZ open spin chain; $q-$Onsager algebra; $q-$vertex operators; Thermodynamic limit

1. Introduction

Inspired by Onsager's strategy for the solution of the two-dimensional Ising model [Ons] and later works on the superintegrable chiral Potts model [vGR, PeAMT, Dav], as well as the conformal field theory program [KBPZ], a new approach to the study of quantum integrable systems has been initiated in [Bas1, Bas2, BK1]. It is based on the explicit formulation of mutally commuting conserved quantities - for instance the Hamiltonian - in terms of the elements of a non-Abelian infinite dimensional algebra. In these works, the non-Abelian algebra considered was the so-called $q-$Onsager algebra which Abelian subalgebra generates the so-called $q-$Dolan-Grady hierarchy. Actually, these objects were exhibited by considering in details the structure of Sklyanin's operator that appears in the standard formulation of models with boundaries[1] [Sk]. In this framework, finding a solution of a specific model then relies on a detailed analysis of the finite or infinite dimensional representations involved. In [BK1, BK2, BK3], this approach was considered in details for the finite XXZ open spin chain with *generic* integrable boundary conditions, in which case known results[2] were interpretated within the representation theory of the $q-$Onsager algebra. There, the spectral problem of the Hamiltonian was studied using certain properties of the $q-$Onsager algebra, especially those related with the concept of tridiagonal pairs [Ter2]. Note, however, that some other properties exhibited in [BK1] - namely the existence of Davies's type of linear relations used to solve the Ising and superintegrable Potts models [Dav] - have not been explored yet. Also, the special case of *diagonal* boundary conditions was not considered within Onsager's approach.

Besides above observations that there is still much room to be explored in the finite size case, it is tempting to study in details the thermodynamic limit of lattice models - for instance the XXZ open spin chain - from this new angle. Several experiences motivate to consider this problem further: First, much is known for the special case of the half-infinite XXZ spin chain with *diagonal* boundary conditions: the spectral problem and the calculation of correlation functions have been studied in details either within the vertex operator approach in [JKKKM] or the algebraic Bethe ansatz approach in [KKMNST]. Then, known results can serve as a straightforward check of the ones that may be obtained within the new approach. Secondly, nothing is known about the correlation functions of local operators in the model with a *non-diagonal* boundary. Indeed, the algebraic Bethe ansatz approach suffer from various technical difficulties. As we will see here, the construction of vacuum vectors for $-1 < q < 0$ using the representation theory of the current algebra in this case is reduced to a well-defined problem. Third, we expect that the Onsager's formulation

[1]This does not imply that the approach solely applies to models with boundaries: for $q = 1$, the Ising [Ons] and superintegrable chiral Potts models [PeAMT, Dav, vGR] are explicit counterexamples of this idea (see also [Ar, AS]).

[2]For instance, the linear relations between the left and right boundary parameters that arise in the Bethe ansatz approach in order to construct a suitable reference state or in the diagonalization of the $Q-$Baxter operator [bXXZ, MMR, BCRS].

exhibits some remarkable properties for special choice of the boundary parameters, for instance the ones corresponding to the thermodynamic limit of the $U_q(sl_2)$-symmetric XXZ spin chain [PS].

Having explained the main motivations, the purpose of this paper is to give first insights on the Onsager's approach in the thermodynamic limit of lattice models as well as practical applications. In particular, a detailed analysis on the exact solution - spectrum and eigenstates - of the half-infinite XXZ open spin chain with diagonal boundary conditions as well as upper (or lower) non-diagonal boundary conditions within the Onsager's approach is presented. Before starting, let us make a few comments on some mathematical aspects that are involved in the process. Indeed, it is important to recall that an Onsager's formulation of the half-infinite XXZ open spin chain previously requires to identify and study the underlying current algebra which first modes generate the so-called $q-$Onsager algebra or augmented $q-$Onsager algebra (see Section 2,3). In this direction, essential steps were bypassed in [BSh, BB2]. In [BSh], a new type of current algebra was proposed which generates the spectrum generating algebra associated with the XXZ open spin chain. In [BB2], a realization of the currents in terms of operators satisfying a Faddeev-Zamomodchikov algebra was obtained, offering a straightforward bosonization scheme for the currents for $-1 < q < 0$. Although recalled in the present paper, we refer the reader to these works for details.

This paper is organized as follows: In Section 2, the Onsager's presentation of the finite size XXZ open chain is described in details: the transfer matrix is explicitly written in terms of the elements of a spectrum generating algebra associated either with diagonal or non-diagonal boundary conditions. In particular, the case of diagonal boundary conditions here presented completes the formulation of [BK3]. Elements of the spectrum generating algebras are explicitely reported in Appendix A, which first ones are shown to generate the $q-$Onsager and augmented $q-$Onsager algebra according to the choice of boundary conditions. In Section 3, the thermodynamic limit of the model is considered: the transfer matrix of the half-infinite XXZ open spin chain is simply expressed in terms of $O_q(\widehat{sl_2})$ currents for q generic. By analogy with Section 2, the properties of the first modes are considered in relation with two different coideal subalgebras of $U_q(\widehat{sl_2})$. Based on these, for $-1 < q < 0$ level one infinite dimensional representations of $O_q(\widehat{sl_2})$ are proposed in Section 4, where explicit expressions for the currents in terms of $U_q(\widehat{sl_2})$ $q-$vertex operators independently confirm the proposal of [BB2]. In Section 5, by analogy with the strategy applied in [BK3] the spectral problem for two of the $O_q(\widehat{sl_2})$ currents is considered. Using this result, the diagonalization of the transfer matrix is considered in details: for the case of diagonal boundary conditions, known results of Jimbo *et al.* [JKKKM] are interpreted in light of the representation theory of the $O_q(\widehat{sl_2})$ current algebra. Then, the case of upper (or lower) non-diagonal boundary conditions is studied: whereas the spectrum is identical to the one for the diagonal case, the eigenstates are written as an infinite sum. Few comments are added in the last Section. For instance, it is shown that the $q-$Onsager algebra or augmented $q-$Onsager algebra emerge as the non-Abelian symmetry of the thermodynamic limit of the XXZ open spin chain depending on the boundary conditions. Although the existence of an infinite dimensional non-Abelian symmetry was definitely expected for both types of boundary conditions, here this issue is clarified. Also, we point out some interesting phenomena for the special boundary conditions chosen in [PS].

Notation . *In this paper, we introduce the $q-$commutator $\big[X,Y\big]_q = qXY - q^{-1}YX$ where q is the deformation parameter, assumed not to be a root of unity.*

2. Alternative presentations of the finite XXZ open spin chain

The finite size XXZ open spin chain with general integrable boundary conditions is the subject of numerous investigations in recent years. Starting from Sklyanin's work [Sk] for the special case of diagonal boundary conditions, it has been later on studied for *generic* or *special* (left-right related) boundary conditions and q (root of unity) [bXXZ, BCRS, Gal, N]. For general integrable boundary conditions and q, its Hamiltonian is given by:

$$
\begin{aligned}
(2.1)\quad H^{(N)}_{XXZ} \;=\; & \sum_{k=1}^{N-1}\Big(\sigma_1^{k+1}\sigma_1^k+\sigma_2^{k+1}\sigma_2^k+\Delta\sigma_3^{k+1}\sigma_3^k\Big)+\frac{(q-q^{-1})}{2}\frac{(\epsilon_+-\epsilon_-)}{(\epsilon_++\epsilon_-)}\sigma_3^1+\frac{2}{(\epsilon_++\epsilon_-)}\big(k_+\sigma_+^1+k_-\sigma_-^1\big)\\
&+\frac{(q-q^{-1})}{2}\frac{(\bar\epsilon_+-\bar\epsilon_-)}{(\bar\epsilon_++\bar\epsilon_-)}\sigma_3^N+\frac{2}{(\bar\epsilon_++\bar\epsilon_-)}\big(\bar k_+\sigma_+^N+\bar k_-\sigma_-^N\big)\ ,
\end{aligned}
$$

where $\sigma_{1,2,3}$ and $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$ are usual Pauli matrices. Here, $\Delta = (q+q^{-1})/2$ denotes the anisotropy parameter. Restricting the parameters to special values or certain relations, one obtains the cases considered in [ABBBQ, Sk, PS, bXXZ, NRGP, Do, KKMNST, BCRS].

In the literature, the most standard presentation of the XXZ open spin chain is based on a generalization of the quantum inverse problem to integrable systems with boundaries. In this approach, starting from an $R-$matrix acting on a finite dimensional representation, Hamiltonians of quantum integrable models are basically generated from solutions $K_-(\zeta)$, $K_+(\zeta)$ of the reflection and dual reflection equations, respectively [Sk]. In this standard presentation, the transfer matrix associated with the XXZ open spin chain can be written as:

$$t^{(N)}(\zeta) = \frac{(-1)^N}{(\zeta^2+\zeta^{-2}-q^2-q^{-2})^N}\ tr_0\Big[K_+(\zeta)\bar{R}_{0\mathtt{N}}(\zeta)\cdots\bar{R}_{0\mathtt{1}}(\zeta)K_-(\zeta)\bar{R}_{0\mathtt{1}}(\zeta)\cdots\bar{R}_{0\mathtt{N}}(\zeta)\Big]\ , \tag{2.2}$$

where tr_0 denotes the trace over the two-dimensional auxiliary space,

$$\bar{R}(\zeta) = \begin{pmatrix} \zeta q - \zeta^{-1}q^{-1} & 0 & 0 & 0 \\ 0 & \zeta-\zeta^{-1} & q-q^{-1} & 0 \\ 0 & q-q^{-1} & \zeta-\zeta^{-1} & 0 \\ 0 & 0 & 0 & \zeta q-\zeta^{-1}q^{-1} \end{pmatrix}\ , \tag{2.3}$$

and the most general elements[3] $K_\pm(\zeta)$ with $c-$number entries take the form

$$K_-(\zeta) \;=\; \begin{pmatrix} \zeta\epsilon_+ + \zeta^{-1}\epsilon_- & k_+(\zeta^2-\zeta^{-2})/(q-q^{-1}) \\ k_-(\zeta^2-\zeta^{-2})/(q-q^{-1}) & \zeta\epsilon_- + \zeta^{-1}\epsilon_+ \end{pmatrix}\ , \tag{2.4}$$

$$K_+(\zeta) \;=\; \begin{pmatrix} q\zeta\bar{\epsilon}_+ + q^{-1}\zeta^{-1}\bar{\epsilon}_- & \bar{k}_+(q^2\zeta^2-q^{-2}\zeta^{-2})/(q-q^{-1}) \\ \bar{k}_-(q^2\zeta^2-q^{-2}\zeta^{-2})/(q-q^{-1}) & q\zeta\bar{\epsilon}_- + q^{-1}\zeta^{-1}\bar{\epsilon}_+ \end{pmatrix}\ . \tag{2.5}$$

In this formulation, the Hamiltonian of the XXZ open spin chain with general integrable boundary conditions (2.1) is obtained as follows[4]:

$$\frac{d}{d\zeta}ln(t^{(N)}(\zeta))|_{\zeta=1} = \frac{2}{(q-q^{-1})}H^{(N)}_{XXZ} + \Big(\frac{(q-q^{-1})}{(q+q^{-1})} + \frac{2N}{(q-q^{-1})}\Delta\Big)I\!I^{(N)}\ . \tag{2.6}$$

More generally, higher mutually commuting local conserved quantities, say H_n with $H_1 \equiv H^{(N)}_{XXZ}$, can be derived similarly by taking higher derivatives of the transfer matrix (2.7).

An alternative presentation of the XXZ open spin chain has been proposed in [BK2]. It is inspired by the strategy developed by Onsager for the two-dimensional Ising model [Ons], later works on the superintegrable chiral Potts and XY models [vGR, PeAMT, Dav, Ar] (see also [AS]) and the vertex operators approach [DFJMN, JKKKM, Ko]: starting from the spectrum generating algebra or hidden non-Abelian algebra symmetry of a quantum integrable model, one is looking for the solution of the model solely using the representation theory of this algebra. In particular, such type of approach applies to the XXZ open spin chain which integrability condition can be associated with a $q-$deformed analog of the Onsager algebra for *generic* boundary conditions [BK2]: it was shown that the transfer matrix can be written in terms of mutually commuting quantities $\mathcal{I}^{(N)}_{2k+1}$ that generate a $q-$deformed analog of the Onsager-Dolan-Grady's hierarchy[5]. Namely,

$$t^{(N)}_{gen-gen}(\zeta) = \sum_{k=0}^{N-1}\mathcal{F}_{2k+1}(\zeta)\ \mathcal{I}^{(N)}_{2k+1} + \mathcal{F}_0(\zeta)\ I\!I^{(N)} \quad \text{with} \qquad \big[\mathcal{I}^{(N)}_{2k+1},\mathcal{I}^{(N)}_{2l+1}\big]=0 \tag{2.7}$$

for all $k,l \in 0,...,N-1$ where

$$\mathcal{I}^{(N)}_{2k+1} = \bar{\epsilon}_+\mathcal{W}^{(N)}_{-k} + \bar{\epsilon}_-\mathcal{W}^{(N)}_{k+1} + \frac{1}{q^2-q^{-2}}\Big(\frac{\bar{k}_-}{k_-}\mathcal{G}^{(N)}_{k+1} + \frac{\bar{k}_+}{k_+}\tilde{\mathcal{G}}^{(N)}_{k+1}\Big) \tag{2.8}$$

---

[3]Note that $K_+(\zeta) = -K^t_-(-\zeta^{-1}q^{-1})|_{\epsilon_\pm\rightarrow\bar{\epsilon}_\mp;k_\pm\rightarrow\bar{k}_\mp}$.

[4]The identity operator acting on N sites is denoted $I\!I^{(N)} = I\!I\otimes\cdots\otimes I\!I$.

[5]The Onsager's (also called Dolan-Grady) hierarchy is an Abelian algebra with elements of the form $I_{2n+1} = \bar{\epsilon}_+(A_n+A_{-n}) + \bar{\epsilon}_-(A_{n+1}+A_{-n+1}) + \kappa G_{n+1}$ with $n\in\mathbb{Z}_+$ generated from the Onsager algebra with defining relations $\big[A_n,A_m\big] = 4G_{n-m}$, $\big[G_m,A_n\big] = 2A_{n+m} - 2A_{n-m}$ and $\big[G_n,G_m\big]=0$ for any $n,m\in\mathbb{Z}$.

and $\mathcal{F}_{2k+1}(\zeta)$ are Laurent polynomials (see [BK2] for details). Note that the parameters $\epsilon_\pm$ of the right boundary - which do not appear explicitely in above formula - are actually hidden in the definition of the elements $\mathcal{W}_{-k}^{(N)}, \mathcal{W}_{k+1}^{(N)}, \mathcal{G}_{k+1}^{(N)}, \tilde{\mathcal{G}}_{k+1}^{(N)}$ which explicit expressions are recalled in Appendix A.

Remark 1. *Another Onsager's presentation can be alternatively considered , in which case the elements of the spectrum generating algebra contain the parameters $\bar{\epsilon}_\pm, \bar{k}_\pm$ of the left boundary. In this case, one substitutes in (2.7)*

$$\mathcal{I}_{2k+1}^{(N)} \rightarrow \overline{\mathcal{I}}_{2k+1}^{(N)} \quad where \quad \overline{\mathcal{I}}_{2k+1}^{(N)} = \epsilon_+ \overline{\mathcal{W}}_{k+1}^{(N)} + \epsilon_- \overline{\mathcal{W}}_{-k}^{(N)} + \frac{1}{q^2-q^{-2}}\Big(\frac{k_-}{\bar{k}_-}\overline{\tilde{\mathcal{G}}}_{k+1}^{(N)} + \frac{k_+}{\bar{k}_+}\overline{\mathcal{G}}_{k+1}^{(N)}\Big) \ . \tag{2.9}$$

The functions $\overline{\mathcal{F}}_{2k+1}(\zeta), \overline{\mathcal{F}}_0(\zeta)$ can be derived following [BK2]*. Here, the elements are given by:*

$$\overline{\mathcal{W}}_{-k}^{(N)} = \Pi_N\big(\mathcal{W}_{k+1}^{(N)}\big)|_{\epsilon_\pm\rightarrow\bar{\epsilon}_\pm; k_\pm\rightarrow\ \bar{k}_\pm} \ , \quad \overline{\mathcal{W}}_{k+1}^{(N)} = \Pi_N\big(\mathcal{W}_{-k}^{(N)}\big)|_{\epsilon_\pm\rightarrow\bar{\epsilon}_\pm; k_\pm\rightarrow\ \bar{k}_\pm} \ ,$$
$$\overline{\mathcal{G}}_{k+1}^{(N)} = \Pi_N\big(\tilde{\mathcal{G}}_{k+1}^{(N)}\big)|_{\epsilon_\pm\rightarrow\bar{\epsilon}_\pm; k_\pm\rightarrow\ \bar{k}_\pm} \ , \quad \overline{\tilde{\mathcal{G}}}_{k+1}^{(N)} = \Pi_N\big(\mathcal{G}_{k+1}^{(N)}\big)|_{\epsilon_\pm\rightarrow\bar{\epsilon}_\pm; k_\pm\rightarrow\ \bar{k}_\pm} \ ,$$

where the permutation operator $\Pi_N\big(a_1\otimes a_2\otimes ...\otimes a_N\big) = a_N\otimes ...\otimes a_2\otimes a_1$ is used.

Having exhibited an Onsager's type of presentation (2.7) for the XXZ open spin chain with *generic* boundary conditions, a natural question is weither such presentation also exists for special boundary conditions that have been considered in the literature. Actually, for special *right diagonal* boundary conditions it is the case provided certain changes in the definition of the basic objects: Following the analysis of [BK2], it is easy to show that the *diagonal* analog of the $q-$Dolan-Grady hierarchy is associated with an Abelian subalgebra generated by the elements $\mathcal{J}_{2l+1}^{(N)}$ such that

$$\mathcal{J}_{2k+1}^{(N)} = \bar{\epsilon}_+ \mathcal{K}_{-k}^{(N)} + \bar{\epsilon}_- \mathcal{K}_{k+1}^{(N)} + \frac{1}{q^2-q^{-2}}\Big(\bar{k}_-\mathcal{Z}_{k+1}^{(N)} + \bar{k}_+\tilde{\mathcal{Z}}_{k+1}^{(N)}\Big) \tag{2.10}$$

where the explicit expressions of $\mathcal{K}_{-k}^{(N)}, \mathcal{K}_{k+1}^{(N)}, \mathcal{Z}_{k+1}^{(N)}, \tilde{\mathcal{Z}}_{k+1}^{(N)}$ are reported in Appendix A. In this special case, the transfer matrix associated with the Hamiltonian (2.1) for $k_\pm \equiv 0$ takes the form:

$$t_{gen-diag}^{(N)}(\zeta) = \sum_{k=0}^{N-1} \mathcal{F}_{2k+1}^{diag}(\zeta)\ \mathcal{J}_{2k+1}^{(N)} + \mathcal{F}_0^{diag}(\zeta)\ I\!I^{(N)} \quad \text{with} \quad \big[\mathcal{J}_{2k+1}^{(N)}, \mathcal{J}_{2l+1}^{(N)}\big] = 0 \ . \tag{2.11}$$

Remark 2. *For special left diagonal (but right generic) boundary conditions, the transfer matrix $t_{diag-gen}^{(N)}(\zeta)$ can be aternatively presented in terms of*

$$\overline{\mathcal{J}}_{2k+1}^{(N)} = \epsilon_+ \overline{\mathcal{K}}_{k+1}^{(N)} + \epsilon_- \overline{\mathcal{K}}_{-k}^{(N)} + \frac{1}{q^2-q^{-2}}\Big(k_-\overline{\tilde{\mathcal{Z}}}_{k+1}^{(N)} + k_+\overline{\mathcal{Z}}_{k+1}^{(N)}\Big) \tag{2.12}$$

where

$$\overline{\mathcal{K}}_{-k}^{(N)} = \Pi_N\big(\mathcal{K}_{k+1}^{(N)}\big)|_{\epsilon_\pm\rightarrow\bar{\epsilon}_\pm} \ , \quad \overline{\mathcal{K}}_{k+1}^{(N)} = \Pi_N\big(\mathcal{K}_{-k}^{(N)}\big)|_{\epsilon_\pm\rightarrow\bar{\epsilon}_\pm} \ ,$$
$$\overline{\mathcal{Z}}_{k+1}^{(N)} = \Pi_N\big(\tilde{\mathcal{Z}}_{k+1}^{(N)}\big)|_{\epsilon_\pm\rightarrow\bar{\epsilon}_\pm} \ , \quad \overline{\tilde{\mathcal{Z}}}_{k+1}^{(N)} = \Pi_N\big(\mathcal{Z}_{k+1}^{(N)}\big)|_{\epsilon_\pm\rightarrow\bar{\epsilon}_\pm} \ .$$

As a consequence, the transfer matrix $t_{diag-diag}^{(N)}(\zeta)$ for the special case of *left and right diagonal* boundary conditions - the simplest case studied in [Sk] using the presentation (2.2) - admits an Onsager's type of presentation, where mutually commuting quantities are simply given by (2.10) with $\bar{k}_\pm \equiv 0$ or, alternatively, (2.12) with $k_\pm \equiv 0$.

For each type of boundary conditions, let us now describe some basic aspects of the corresponding spectrum generating algebras. According to the choice of boundary conditions associated with the left and right side of the spin chain, two different types of spectrum generating algebras arise in Onsager's presentation of the XXZ open spin chain. For *generic* boundary conditions, it is the infinite dimensional algebra called $\mathcal{A}_q$ which ensures the integrability of the model (2.1). It has been studied in details in [BK1, BK2], where the elements $\mathcal{W}_{-k}^{(N)}, \mathcal{W}_{k+1}^{(N)}, \mathcal{G}_{k+1}^{(N)}, \tilde{\mathcal{G}}_{k+1}^{(N)}$ are known to

satisfy, in particular, the subset of relations (see also [BSh])

$$\big[\mathsf{W}_0,\mathsf{W}_{k+1}\big]=\big[\mathsf{W}_{-k},\mathcal{W}_1\big]=\frac{1}{(q+q^{-1})}\big(\tilde{\mathsf{G}}_{k+1}-\mathsf{G}_{k+1}\big)\ ,\quad \big[\mathsf{W}_0,\mathsf{W}_{-k}\big]=0\ ,\quad \big[\mathsf{W}_1,\mathsf{W}_{k+1}\big]=0\ ,$$
$$\big[\mathsf{W}_0,\mathsf{G}_{k+1}\big]_q=\big[\tilde{\mathsf{G}}_{k+1},\mathsf{W}_0\big]_q=\rho\mathsf{W}_{-k-1}-\rho\mathsf{W}_{k+1}\ ,$$
$$\big[\mathsf{G}_{k+1},\mathsf{W}_1\big]_q=\big[\mathsf{W}_1,\tilde{\mathsf{G}}_{k+1}\big]_q=\rho\mathsf{W}_{k+2}-\rho\mathsf{W}_{-k}\ , \tag{2.13}$$

provided the identification

$$\rho=(q+q^{-1})^2k_+k_-\ . \tag{2.14}$$

Remarkably, according to the explicit expressions of the first elements for *generic* values of $\epsilon_\pm,k_\pm,q$,

$$\begin{aligned}\mathcal{W}_0^{(N)}&=&(k_+\sigma_++k_-\sigma_-)\otimes I\!I^{(N-1)}+q^{\sigma_3}\otimes\mathcal{W}_0^{(N-1)}\ ,\quad \mathcal{W}_0^{(0)}=\epsilon_+\ ,\\ \mathcal{W}_1^{(N)}&=&(k_+\sigma_++k_-\sigma_-)\otimes I\!I^{(N-1)}+q^{-\sigma_3}\otimes\mathcal{W}_1^{(N-1)}\ ,\quad \mathcal{W}_1^{(0)}=\epsilon_-\ ,\end{aligned} \tag{2.15}$$

one observes[6] [BK2] that $\mathcal{A}_q$ admits another presentation in terms of the so-called $q-$*Onsager algebra* with defining ($q-$Dolan-Grady) relations [Ter2]:

$$\begin{aligned}\big[\mathcal{W}_0^{(N)},\big[\mathcal{W}_0^{(N)},\big[\mathcal{W}_0^{(N)},\mathcal{W}_1^{(N)}\big]_q\big]_{q^{-1}}\big]&=&\rho\big[\mathcal{W}_0^{(N)},\mathcal{W}_1^{(N)}\big]\ ,\\ \big[\mathcal{W}_1^{(N)},\big[\mathcal{W}_1^{(N)},\big[\mathcal{W}_1^{(N)},\mathcal{W}_0^{(N)}\big]_q\big]_{q^{-1}}\big]&=&\rho\big[\mathcal{W}_1^{(N)},\mathcal{W}_0^{(N)}\big]\ .\end{aligned} \tag{2.16}$$

Note that these relations are a special case of the defining relations of the tridiagonal algebras [Ter2].

By analogy, the defining relations of the infinite dimensional algebra $\mathcal{A}_q^{diag}$ satisfied by the elements $\mathcal{K}_{-k}^{(N)}$, $\mathcal{K}_{k+1}^{(N)}$, $\mathcal{Z}_{k+1}^{(N)}$, $\tilde{\mathcal{Z}}_{k+1}^{(N)}$ which ensures the integrability of the model (2.1) for *right diagonal* boundary conditions can be studied along the same line using the technique considered in [BK1, BK2]. Obviously, they can be directly obtained from the results of [BK2] for the special case $k_\pm\equiv0$. Similarly to the case of generic boundary conditions, for our purpose it will be sufficient to focus on the set of relations satisfied by the first elements which explicit expressions are given by (see Appendix A):

$$\begin{aligned}\mathcal{K}_0^{(N)}&=&q^{\sigma_3}\otimes\mathcal{K}_0^{(N-1)}\ ,\quad \mathcal{K}_0^{(0)}=\epsilon_+\ ,\\ \mathcal{K}_1^{(N)}&=&q^{-\sigma_3}\otimes\mathcal{K}_1^{(N-1)}\ ,\quad \mathcal{K}_1^{(0)}=\epsilon_-\ ,\\ \mathcal{Z}_1^{(N)}&=&I\!I\otimes\mathcal{Z}_1^{(N-1)}+(q^2-q^{-2})\sigma_-\otimes\Big(\mathcal{K}_0^{(N-1)}+\mathcal{K}_1^{(N-1)}\Big)\ ,\\ \tilde{\mathcal{Z}}_1^{(N)}&=&I\!I\otimes\tilde{\mathcal{Z}}_1^{(N-1)}+(q^2-q^{-2})\sigma_+\otimes\Big(\mathcal{K}_0^{(N-1)}+\mathcal{K}_1^{(N-1)}\Big)\ ,\quad \mathcal{Z}_1^{(0)}=\tilde{\mathcal{Z}}_1^{(0)}=0\ .\end{aligned} \tag{2.17}$$

By straightforward calculations, for generic values of $\epsilon_\pm,q$, the elements $\mathcal{K}_0^{(N)},\mathcal{K}_1^{(N)},\mathcal{Z}_1^{(N)},\tilde{\mathcal{Z}}_1^{(N)}$ are found to generate the *augmented* $q-$*Onsager algebra* with defining relations:

$$\begin{aligned}\big[\mathcal{K}_0^{(N)},\mathcal{K}_1^{(N)}\big]&=&0\ ,\\ \mathcal{K}_0^{(N)}\mathcal{Z}_1^{(N)}&=&q^{-2}\mathcal{Z}_1^{(N)}\mathcal{K}_0^{(N)}\ ,\qquad \mathcal{K}_0^{(N)}\tilde{\mathcal{Z}}_1^{(N)}=q^2\tilde{\mathcal{Z}}_1^{(N)}\mathcal{K}_0^{(N)}\ ,\\ \mathcal{K}_1^{(N)}\mathcal{Z}_1^{(N)}&=&q^{2}\mathcal{Z}_1^{(N)}\mathcal{K}_1^{(N)}\ ,\qquad \mathcal{K}_1^{(N)}\tilde{\mathcal{Z}}_1^{(N)}=q^{-2}\tilde{\mathcal{Z}}_1^{(N)}\mathcal{K}_1^{(N)}\ ,\\ \big[\mathcal{Z}_1^{(N)},\big[\mathcal{Z}_1^{(N)},\big[\mathcal{Z}_1^{(N)},\tilde{\mathcal{Z}}_1^{(N)}\big]_q\big]_{q^{-1}}\big]&=&\rho_{diag}\mathcal{Z}_1^{(N)}(\mathcal{K}_1^{(N)}\mathcal{K}_1^{(N)}-\mathcal{K}_0^{(N)}\mathcal{K}_0^{(N)})\mathcal{Z}_1^{(N)},\\ \big[\tilde{\mathcal{Z}}_1^{(N)},\big[\tilde{\mathcal{Z}}_1^{(N)},\big[\tilde{\mathcal{Z}}_1^{(N)},\mathcal{Z}_1^{(N)}\big]_q\big]_{q^{-1}}\big]&=&\rho_{diag}\tilde{\mathcal{Z}}_1^{(N)}(\mathcal{K}_0^{(N)}\mathcal{K}_0^{(N)}-\mathcal{K}_1^{(N)}\mathcal{K}_1^{(N)})\tilde{\mathcal{Z}}_1^{(N)}\end{aligned} \tag{2.18}$$

with

$$\rho_{diag}=\frac{(q^3-q^{-3})(q^2-q^{-2})^3}{q-q^{-1}}\ . \tag{2.19}$$

[6]More generally, all higher elements can be written as polynomials in $\mathcal{W}_0^{(N)},\mathcal{W}_1^{(N)}$ [BB3]. For instance, $\mathcal{G}_1^{(N)}=I\!I\otimes\mathcal{G}_1^{(N-1)}+(q^2-q^{-2})k_-\sigma_-\otimes\Big(\mathcal{W}_0^{(N-1)}+\mathcal{W}_1^{(N-1)}\Big)+k_+k_-(q-q^{-1})I\!I^{(N)}=[\mathcal{W}_1^{(N)},\mathcal{W}_0^{(N)}]_q$. Similarly, one has $\tilde{\mathcal{G}}_1^{(N)}=[\mathcal{W}_0^{(N)},\mathcal{W}_1^{(N)}]_q$.

Note that the augmented $q-$Onsager algebra is a special case of the augmented tridiagonal algebra [IT], which finite dimensional representations for q not a root of unity have been classified in [IT].

Alternatively, the presentations (2.9) and (2.12) may be considered: the analysis above and the definition of Π_N implies that the elements $\overline{\mathcal{W}}_{-k}^{(N)}, \overline{\mathcal{W}}_{k+1}^{(N)}, \overline{\mathcal{G}}_{k+1}^{(N)}, \overline{\tilde{\mathcal{G}}}_{k+1}^{(N)}$ and $\overline{\mathcal{K}}_{-k}^{(N)}, \overline{\mathcal{K}}_{k+1}^{(N)}, \overline{\mathcal{Z}}_{k+1}^{(N)}, \overline{\tilde{\mathcal{Z}}}_{k+1}^{(N)}$ generate the algebras $\mathcal{A}_q$ and $\mathcal{A}_q^{diag}$, respectively. In particular, the elements:

$$(2.20)\qquad \begin{aligned} \overline{\mathcal{W}}_0^{(N)} &= I\!I^{(N-1)} \otimes (\bar{k}_+\sigma_+ + \bar{k}_-\sigma_-) + \overline{\mathcal{W}}_0^{(N-1)} \otimes q^{-\sigma_3}\ , \quad \overline{\mathcal{W}}_0^{(0)} = \bar{\epsilon}_-\ , \\ \overline{\mathcal{W}}_1^{(N)} &= I\!I^{(N-1)} \otimes (\bar{k}_+\sigma_+ + \bar{k}_-\sigma_-) + \overline{\mathcal{W}}_1^{(N-1)} \otimes q^{\sigma_3}\ , \quad \overline{\mathcal{W}}_1^{(0)} = \bar{\epsilon}_+ \end{aligned}$$

satisfy the $q-$Onsager algebra relations (2.16) with $\rho = (q+q^{-1})^2\bar{k}_+\bar{k}_-$.

On the other hand, the elements:

$$(2.21)\qquad \begin{aligned} \overline{\mathcal{K}}_0^{(N)} &= \overline{\mathcal{K}}_0^{(N-1)} \otimes q^{-\sigma_3}\ , \quad \overline{\mathcal{K}}_0^{(0)} = \bar{\epsilon}_-\ , \\ \overline{\mathcal{K}}_1^{(N)} &= \overline{\mathcal{K}}_1^{(N-1)} \otimes q^{\sigma_3}\ , \quad \overline{\mathcal{K}}_1^{(0)} = \bar{\epsilon}_+\ , \\ \overline{\mathcal{Z}}_1^{(N)} &= \overline{\mathcal{Z}}_1^{(N-1)} \otimes I\!I + (q^2-q^{-2})\left(\overline{\mathcal{K}}_0^{(N-1)} + \overline{\mathcal{K}}_1^{(N-1)}\right) \otimes \sigma_+\ , \\ \overline{\tilde{\mathcal{Z}}}_1^{(N)} &= \overline{\tilde{\mathcal{Z}}}_1^{(N-1)} \otimes I\!I + (q^2-q^{-2})\left(\overline{\mathcal{K}}_0^{(N-1)} + \overline{\mathcal{K}}_1^{(N-1)}\right) \otimes \sigma_-\ , \quad \overline{\mathcal{Z}}_1^{(0)} = \overline{\tilde{\mathcal{Z}}}_1^{(0)} = 0 \end{aligned}$$

satisfy the augmented $q-$Onsager algebra with relations (2.18) and (2.19).

To resume, recall that the explicit relation between Sklyanin's presentation (2.2) [Sk] and Onsager's type of presentation (2.7) of the XXZ open spin chain with generic boundary conditions has been described in details in [BK2] (see also [BK1]). Above results for the special case of right diagonal, left diagonal or right and left diagonal boundary conditions complete the correspondence. They are collected in the following table, where the set of integrals of motions (IMs) are specified according to the presentation chosen:

open XXZ chain	Spectrum gen. algebra	IMs (1st presentation)	IMs (2nd presentation)
right-left generic bcs.	$\mathcal{A}_q \to q-$Onsager	$\mathcal{I}_{2k+1}^{(N)}$	$\overline{\mathcal{I}}_{2k+1}^{(N)}$
right diag. bcs. $k_\pm = 0$	$\mathcal{A}_q^{diag} \to$ aug. $q-$Onsager *or* $\mathcal{A}_q \to q-$Onsager	$\mathcal{J}_{2k+1}^{(N)}$	$\overline{\mathcal{I}}_{2k+1}^{(N)}\vert_{k_\pm=0}$
left diag. bcs. $\bar{k}_\pm = 0$	$\mathcal{A}_q \to q-$Onsager *or* $\mathcal{A}_q^{diag} \to$ aug. $q-$Onsager	$\mathcal{I}_{2k+1}^{(N)}\vert_{\bar{k}_\pm=0}$	$\overline{\mathcal{J}}_{2k+1}^{(N)}$
right-left diag. bcs. $\bar{k}_\pm = k_\pm = 0$	$\mathcal{A}_q^{diag} \to$ aug. $q-$Onsager	$\mathcal{J}_{2k+1}^{(N)}\vert_{\bar{k}_\pm=0}$	$\overline{\mathcal{J}}_{2k+1}^{(N)}\vert_{k_\pm=0}$

Note that the spectrum of the XXZ open spin chain Hamiltonian (2.1) with *right* diagonal boundary conditions ($k_\pm = 0$) and *left* diagonal boundary ($\bar{k}_\pm = 0$) conditions may be considered using the properties either of $\mathcal{A}_q$ or $\mathcal{A}_q^{diag}$. Also, it is important to stress that the list of cases presented above is not exhaustive: for instance, one may consider the set of diagonal boundary conditions $k_\pm = \bar{k}_\pm = 0$, $\epsilon_+ \neq 0$, $\bar{\epsilon}_- \neq 0$ and $\epsilon_- = \bar{\epsilon}_+ = 0$ discussed in [ABBBQ, PS]. In this special case, let us remark that the defining relations satisfied by the fundamental elements of the corresponding 'larger' spectrum generating algebra can be derived in a straighforward manner (see a related work [R]). As we will discussed in the last Section, in the thermodynamic limit the diagonalization of the $q-$Dolan-Grady hierarchy in this special case exhibits interesting features.

3. Onsager's presentation: the thermodynamic limit

The half-infinite XXZ open spin chain with an integrable boundary can be considered as the thermodynamic limit $N \to \infty$ of the finite XXZ open spin chain (2.1). Consider the Hamiltonian:

$$(3.1)\quad H_{\frac{1}{2}XXZ} = -\frac{1}{2}\sum_{k=1}^{\infty}\left(\sigma_1^{k+1}\sigma_1^k + \sigma_2^{k+1}\sigma_2^k + \Delta\sigma_3^{k+1}\sigma_3^k\right) - \frac{(q-q^{-1})(\epsilon_+-\epsilon_-)}{4(\epsilon_++\epsilon_-)}\sigma_3^1 - \frac{1}{(\epsilon_++\epsilon_-)}\left(k_+\sigma_+^1 + k_-\sigma_-^1\right)\ .$$

Note that the normalization in front of the Hamiltonian has been changed compared with (2.1), to fit later on with the definitions of [JKKKMW] for the special case of *diagonal* boundary conditions $k_\pm = 0$. By definition, the Hamiltonian formally acts on an infinite dimensional vector space $\mathcal{V}$ which can be written as an infinite tensor product of 2−dimensional $\mathbb{C}^2$ vector space. According to the ordering of the tensor components in (3.1),

$$\mathcal{V} = \cdots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \ . \tag{3.2}$$

A transfer matrix associated with the Hamiltonian (3.1) can be proposed by analogy with the expressions (2.7), (2.11) derived for the finite size case. As we are going to explain, it can be written in terms of the elements of the current algebra associated with $\mathcal{A}_q$ and $\mathcal{A}_q^{diag}$. First, recall that the defining relations associated with $\mathcal{A}_q$ can be derived from the current algebra that has been introduced in [[BSh], Definition 2.2]. With minor changes, the defining relations associated with $\mathcal{A}_q^{diag}$ can be obtained similarly. Actually, both sets of defining relations can be extracted from the same current algebra - denoted $O_q(\widehat{sl_2})$ - using two different homomorphisms (mode expansion) given below. To show this, following [BSh] define the formal variables $U(\zeta) = (q\zeta^2 + q^{-1}\zeta^{-2})/(q + q^{-1})$. Let us introduce the current algebra $O_q(\widehat{sl_2})$ with defining relations:

$$\big[\mathcal{W}_\pm(\zeta), \mathcal{W}_\pm(\xi)\big] = 0 \ , \tag{3.3}$$

$$\big[\mathcal{W}_+(\zeta), \mathcal{W}_-(\xi)\big] + \big[\mathcal{W}_-(\zeta), \mathcal{W}_+(\xi)\big] = 0 \ , \tag{3.4}$$

$$\begin{aligned}
&(U(\zeta) - U(\xi))\big[\mathcal{W}_\pm(\zeta), \mathcal{W}_\mp(\xi)\big] = \frac{(q - q^{-1})}{(q + q^{-1})^3} \left(\mathcal{Z}_\pm(\zeta)\mathcal{Z}_\mp(\xi) - \mathcal{Z}_\pm(\xi)\mathcal{Z}_\mp(\zeta)\right) \ , \\
&\mathcal{W}_\pm(\zeta)\mathcal{W}_\pm(\xi) - \mathcal{W}_\mp(\zeta)\mathcal{W}_\mp(\xi) + \frac{1}{(q^2 - q^{-2})(q + q^{-1})^2}\big[\mathcal{Z}_\pm(\zeta), \mathcal{Z}_\mp(\xi)\big] \\
&\qquad + \frac{1 - U(\zeta)U(\xi)}{U(\zeta) - U(\xi)}\big(\mathcal{W}_\pm(\zeta)\mathcal{W}_\mp(\xi) - \mathcal{W}_\pm(\xi)\mathcal{W}_\mp(\zeta)\big) = 0 \ , \\
&U(\zeta)\big[\mathcal{Z}_\mp(\xi), \mathcal{W}_\pm(\zeta)\big]_q - U(\xi)\big[\mathcal{Z}_\mp(\zeta), \mathcal{W}_\pm(\xi)\big]_q - (q - q^{-1})\big(\mathcal{W}_\mp(\zeta)\mathcal{Z}_\mp(\xi) - \mathcal{W}_\mp(\xi)\mathcal{Z}_\mp(\zeta)\big) = 0 \ , \\
&U(\zeta)\big[\mathcal{W}_\mp(\zeta), \mathcal{Z}_\mp(\xi)\big]_q - U(\xi)\big[\mathcal{W}_\mp(\xi), \mathcal{Z}_\mp(\zeta)\big]_q - (q - q^{-1})\big(\mathcal{W}_\pm(\zeta)\mathcal{Z}_\mp(\xi) - \mathcal{W}_\pm(\xi)\mathcal{Z}_\mp(\zeta)\big) = 0 \ ,
\end{aligned} \tag{3.5}$$

$$\big[\mathcal{Z}_\epsilon(\zeta), \mathcal{W}_\pm(\xi)\big] + \big[\mathcal{W}_\pm(\zeta), \mathcal{Z}_\epsilon(\xi)\big] = 0 \ , \quad \forall \epsilon = \pm \ , \tag{3.6}$$

$$\big[\mathcal{Z}_\pm(\zeta), \mathcal{Z}_\pm(\xi)\big] = 0 \ , \tag{3.7}$$

$$\big[\mathcal{Z}_+(\zeta), \mathcal{Z}_-(\xi)\big] + \big[\mathcal{Z}_-(\zeta), \mathcal{Z}_+(\xi)\big] = 0 \ . \tag{3.8}$$

The homomorphism proposed in [[BSh], Theorem 2] gives the explicit relation between the $O_q(\widehat{sl_2})$ current algebra with defining relations (3.3)-(3.8) and the defining relations of $\mathcal{A}_q$ [[BSh], Definition 3.1]. Namely, for $k_\pm \neq 0$ one considers:

$$\begin{aligned}
\mathcal{W}_+(\zeta) &\rightarrow \sum_{k\in\mathbb{Z}_+} \mathsf{W}_{-k} U(\zeta)^{-k-1} \ , \quad \mathcal{W}_-(\zeta) \rightarrow \sum_{k\in\mathbb{Z}_+} \mathsf{W}_{k+1} U(\zeta)^{-k-1} \ , \\
\mathcal{Z}_+(\zeta) &\rightarrow \frac{1}{k_-}\sum_{k\in\mathbb{Z}_+} \mathsf{G}_{k+1} U(\zeta)^{-k-1} + \frac{k_+(q + q^{-1})^2}{(q - q^{-1})} \ , \\
\mathcal{Z}_-(\zeta) &\rightarrow \frac{1}{k_+}\sum_{k\in\mathbb{Z}_+} \tilde{\mathsf{G}}_{k+1} U(\zeta)^{-k-1} + \frac{k_-(q + q^{-1})^2}{(q - q^{-1})} \ .
\end{aligned} \tag{3.9}$$

By analogy[7], the defining relations of $\mathcal{A}_q^{diag}$ follow from (3.3)-(3.8) by considering instead the homomorphism:

$$\begin{aligned}
\mathcal{W}_+(\zeta) &\rightarrow \sum_{k\in\mathbb{Z}_+} \mathsf{K}_{-k} U(\zeta)^{-k-1} \ , \quad \mathcal{W}_-(\zeta) \rightarrow \sum_{k\in\mathbb{Z}_+} \mathsf{K}_{k+1} U(\zeta)^{-k-1} \ , \\
\mathcal{Z}_+(\zeta) &\rightarrow \sum_{k\in\mathbb{Z}_+} \mathsf{Z}_{k+1} U(\zeta)^{-k-1} \ , \quad \mathcal{Z}_-(\zeta) \rightarrow \sum_{k\in\mathbb{Z}_+} \tilde{\mathsf{Z}}_{k+1} U(\zeta)^{-k-1} \ .
\end{aligned} \tag{3.10}$$

[7] Recall that the relations satisfied by the elements of $\mathcal{A}_q^{diag}$ follow from the ones satisfied by the elements of $\mathcal{A}_q$ by setting $k_\pm = 0$.

Note that according to the results of the previous Section, another homomorphism may be alternatively considered. It is given by $\mathcal{W}_\pm(\zeta) \to \overline{\mathcal{W}}_\pm(\zeta)$, $\mathcal{Z}_\pm(\zeta) \to \overline{\mathcal{Z}}_\pm(\zeta)$ with the following substitutions in the r.h.s of the mode expansions (3.9) and (3.10), respectively:

$$\begin{array}{lllll} \mathsf{W}_{-k} \to \overline{\mathsf{W}}_{-k} \ , & \mathsf{W}_{k+1} \to \overline{\mathsf{W}}_{k+1} \ , & \mathsf{G}_{k+1} \to \overline{\mathsf{G}}_{k+1} \ , & \tilde{\mathsf{G}}_{k+1} \to \overline{\tilde{\mathsf{G}}}_{k+1} \ , & k_\pm \to \bar{k}_\mp \\ \mathsf{K}_{-k} \to \overline{\mathsf{K}}_{-k} \ , & \mathsf{K}_{k+1} \to \overline{\mathsf{K}}_{k+1} \ , & \mathsf{Z}_{k+1} \to \overline{\mathsf{Z}}_{k+1} \ , & \tilde{\mathsf{Z}}_{k+1} \to \overline{\tilde{\mathsf{Z}}}_{k+1} \ . & \end{array}$$

Now, a generating function for all mutually commuting quantities can be built in terms of $O_q(\widehat{sl_2})$ currents that act on $\mathcal{V}$. Inspired by (2.9) and (2.12), we define[8]:

$$\overline{\mathcal{I}}(\zeta) = \epsilon_+ \overline{\mathcal{W}}_-(-\zeta^{-1}q^{-1}) + \epsilon_- \overline{\mathcal{W}}_+(-\zeta^{-1}q^{-1}) + \frac{1}{q^2-q^{-2}}\left(k_- \overline{\mathcal{Z}}_-(-\zeta^{-1}q^{-1}) + k_+ \overline{\mathcal{Z}}_+(-\zeta^{-1}q^{-1})\right) \ . \tag{3.11}$$

The transfer matrix associated with the half-infinite XXZ spin chain (3.1) can now be constructed in terms of $O_q(\widehat{sl_2})$ currents acting on $\mathcal{V}$, inspired by the Onsager's presentation of the finite chain described in the previous Section. By analogy with (2.7), (2.11) together with (2.6) and using above quantities, in the following Sections we will consider:

$$t^{(\mathcal{V})}(\zeta) = g\frac{(\zeta^2-\zeta^{-2})}{\rho(\zeta)}\overline{\mathcal{I}}^{(\mathcal{V})}(\zeta) \qquad \text{and} \qquad \frac{d}{d\zeta}t^{(\mathcal{V})}(\zeta)|_{\zeta=1} = -\frac{4}{(q-q^{-1})}H_{\frac{1}{2}XXZ} \tag{3.12}$$

where the index $(\mathcal{V})$ refers to the space on which the currents act and the function $\rho(\zeta)$ is chosen such that

$$t^{(\mathcal{V})}(\zeta) = t^{(\mathcal{V})}(-\zeta^{-1}q^{-1}) \ , \quad t^{(\mathcal{V})}(\zeta)t^{(\mathcal{V})}(\zeta^{-1}) = \mathrm{id} \ , \quad t^{(\mathcal{V})}(\zeta)|_{\zeta=1} = \mathrm{id} \ .$$

In the case of the infinite XXZ spin chain, recall that the fundamental operators exhibited in [FM, Ji, DFJMN] are identified with Chevalley elements of $U_q(\widehat{sl_2})$ acting on an infinite tensor product of two-dimensional vector spaces, thanks to the coproduct structure. For the half-infinite XXZ spin chain (3.1), the fundamental generators of $\mathcal{A}_q$ and $\mathcal{A}_q^{diag}$ can be written as linear combinations of Chevalley elements of $U_q(\widehat{sl_2})$ acting on $\mathcal{V}$, as we are going to show.

We start by considering the fundamental generators of $\mathcal{A}_q$. Recall that the explicit expressions (2.15) hold for any N. In the thermodynamic limit $N \to \infty$, one has:

$$\begin{aligned} \mathcal{W}_0^{(\infty)} &= \sum_{j=1}^{\infty}\left(... \otimes q^{\sigma_3} \otimes q^{\sigma_3} \otimes \underbrace{(k_+\sigma_+ + k_-\sigma_-)}_{site j} \otimes \mathbb{I} \otimes ... \otimes \mathbb{I}\right) \quad + \quad \epsilon_+ \left(... \otimes q^{\sigma_3} \otimes q^{\sigma_3}\right) \ , \\ \mathcal{W}_1^{(\infty)} &= \sum_{j=1}^{\infty}\left(... \otimes q^{-\sigma_3} \otimes q^{-\sigma_3} \otimes \underbrace{(k_+\sigma_+ + k_-\sigma_-)}_{site j} \otimes \mathbb{I} \otimes ... \otimes \mathbb{I}\right) \quad + \quad \epsilon_- \left(... \otimes q^{-\sigma_3} \otimes q^{-\sigma_3}\right) \ . \end{aligned} \tag{3.13}$$

Following [Bas2], one realizes the elements $\mathsf{W}_0, \mathsf{W}_1$ as linear combinations of Chevalley elements of $U_q(\widehat{sl_2})$. Define

$$\begin{aligned} \mathsf{W}_0 &= k_+ e_1 + k_- q^{-1} f_1 q^{h_1} + \epsilon_+ q^{h_1} \ , \\ \mathsf{W}_1 &= k_- e_0 + k_+ q^{-1} f_0 q^{h_0} + \epsilon_- q^{h_0} \end{aligned} \tag{3.14}$$

which satisfy the defining relations of the $q-$Onsager algebra [Bas2]:

$$\begin{aligned} \big[\mathsf{W}_0, \big[\mathsf{W}_0, \big[\mathsf{W}_0, \mathsf{W}_1\big]_q\big]_{q^{-1}}\big] &= \rho\big[\mathsf{W}_0, \mathsf{W}_1\big] \ , \\ \big[\mathsf{W}_1, \big[\mathsf{W}_1, \big[\mathsf{W}_1, \mathsf{W}_0\big]_q\big]_{q^{-1}}\big] &= \rho\big[\mathsf{W}_1, \mathsf{W}_0\big] \end{aligned} \tag{3.15}$$

with (2.14). The fundamental operators of the half-infinite XXZ open spin chain (3.13) are recovered as follows. For the choice[9] of $U_q(\widehat{sl_2})$ coproduct considered in [DFJMN] (see Appendix B), one introduces the coaction[10] map

[8]Note that the defining relations (3.3)-(3.8) are invariant under the substitution $\zeta \to -\zeta^{-1}q^{-1}$

[9]In [BK2], a different coproduct is considered.

[10]In general, given a Hopf algebra $\mathcal{H}$ with comultiplication Δ and counit $\mathcal{E}$, $\mathcal{I}$ is called a left $\mathcal{H}-$comodule (coideal subalgebra of $\mathcal{H}$) if there exists a coaction map $\delta: \ \mathcal{I} \to \mathcal{H} \otimes \mathcal{I}$ such that (right coaction maps are defined similarly)

$$(\Delta \times id) \circ \delta = (id \times \delta) \circ \delta \ , \qquad (\mathcal{E} \times id) \circ \delta \cong id \ .$$

$\delta: \quad \mathcal{A}_q \rightarrow U_q(\widehat{sl_2}) \otimes \mathcal{A}_q$ defined by:

$$\begin{aligned} \delta(\mathsf{W}_0) &= (k_+ e_1 + k_- q^{-1} f_1 q^{h_1}) \otimes 1 + q^{h_1} \otimes \mathsf{W}_0 \ , \\ \delta(\mathsf{W}_1) &= (k_- e_0 + k_+ q^{-1} f_0 q^{h_0}) \otimes 1 + q^{h_0} \otimes \mathsf{W}_1 \ . \end{aligned} \tag{3.16}$$

Let $\delta^{(N)} = (id \times \delta) \circ \delta^{(N-1)}$. Then, for $N \rightarrow \infty$ it follows that $\delta^{(N)}(\mathsf{W}_0)$ and $\delta^{(N)}(\mathsf{W}_1)$ act as (3.13) on $\mathcal{V}$, respectively.

Alternatively, let us mention that another realization may be considered, that will be useful later on. Namely, the elements

$$\begin{aligned} \overline{\mathsf{W}}_0 &= \bar{k}_+ q^{-1} e_1 q^{-h_1} + \bar{k}_- f_1 + \bar{\epsilon}_- q^{-h_1} \ , \\ \overline{\mathsf{W}}_1 &= \bar{k}_- q^{-1} e_0 q^{-h_0} + \bar{k}_+ f_0 + \bar{\epsilon}_+ q^{-h_0} \end{aligned} \tag{3.17}$$

also satisfy (3.15) with $\rho = (q+q^{-1})^2 \bar{k}_+ \bar{k}_-$. In this case, for the choice of $U_q(\widehat{sl_2})$ coproduct (6.15) the corresponding coaction map $\overline{\delta}: \quad \mathcal{A}_q \rightarrow \mathcal{A}_q \otimes U_q(\widehat{sl_2})$ is such that:

$$\begin{aligned} \overline{\delta}(\overline{\mathsf{W}}_0) &= 1 \otimes (\bar{k}_+ q^{-1} e_1 q^{-h_1} + \bar{k}_- f_1) + \overline{\mathsf{W}}_0 \otimes q^{-h_1} \ , \\ \overline{\delta}(\overline{\mathsf{W}}_1) &= 1 \otimes (\bar{k}_- q^{-1} e_0 q^{-h_0} + \bar{k}_+ f_0) + \overline{\mathsf{W}}_1 \otimes q^{-h_0} \ . \end{aligned} \tag{3.18}$$

We then turn to $\mathcal{A}_q^{diag}$. Using the explicit expressions (2.17), in the thermodynamic limit $N \rightarrow \infty$ the fundamental generators take the form:

$$\mathcal{K}_0^{(\infty)} = \epsilon_+ \big(\ldots \otimes q^{\sigma_3} \otimes q^{\sigma_3} \big) \ , \qquad \mathcal{K}_1^{(\infty)} = \epsilon_- \big(\ldots \otimes q^{-\sigma_3} \otimes q^{-\sigma_3} \big) \ , \tag{3.19}$$

$$\mathcal{Z}_1^{(\infty)} = (q^2 - q^{-2}) \Big(\epsilon_+ \sum_{j=1}^{\infty} \big(\ldots \otimes I\!I \otimes \underbrace{\sigma_-}_{site j} \otimes q^{\sigma_3} \otimes \ldots \otimes q^{\sigma_3} \big) \ + \ \epsilon_- \sum_{j=1}^{\infty} \big(\ldots \otimes I\!I \otimes \underbrace{\sigma_-}_{site j} \otimes q^{-\sigma_3} \otimes \ldots \otimes q^{-\sigma_3} \big) \Big) \ ,$$

$$\tilde{\mathcal{Z}}_1^{(\infty)} = (q^2 - q^{-2}) \Big(\epsilon_+ \sum_{j=1}^{\infty} \big(\ldots \otimes I\!I \otimes \underbrace{\sigma_+}_{site j} \otimes q^{\sigma_3} \otimes \ldots \otimes q^{\sigma_3} \big) \ + \ \epsilon_- \sum_{j=1}^{\infty} \big(\ldots \otimes I\!I \otimes \underbrace{\sigma_+}_{site j} \otimes q^{-\sigma_3} \otimes \ldots \otimes q^{-\sigma_3} \big) \Big) \ .$$

Using (3.14), it is straightforward to extract a realization in terms of $U_q(\widehat{sl_2})$ elements: indeed, note that the fundamental operators $\mathcal{Z}_1^{(\infty)}, \tilde{\mathcal{Z}}_1^{(\infty)}$ can be derived from the $q-$commutators $[\mathcal{W}_1^{(\infty)}, \mathcal{W}_0^{(\infty)}]_q$ and $[\mathcal{W}_0^{(\infty)}, \mathcal{W}_1^{(\infty)}]_q$, respectively, by setting $k_\pm = 0$. The explicit expressions (3.14) then suggest to consider:

$$\begin{aligned} \mathsf{K}_0 &= \epsilon_+ q^{h_1} \ , \qquad \mathsf{K}_1 = \epsilon_- q^{h_0} \ , \\ \mathsf{Z}_1 &= (q^2 - q^{-2}) \big(\epsilon_+ q^{-1} e_0 q^{h_1} + \epsilon_- f_1 q^{h_1 + h_0} \big) \ , \\ \tilde{\mathsf{Z}}_1 &= (q^2 - q^{-2}) \big(\epsilon_- q^{-1} e_1 q^{h_0} + \epsilon_+ f_0 q^{h_1 + h_0} \big) \end{aligned} \tag{3.20}$$

which, as one can check, satisfy an augmented $q-$Onsager algebra with defining relations:

$$\begin{aligned} [\mathsf{K}_0, \mathsf{K}_1] &= 0 \ , \\ \mathsf{K}_0 \mathsf{Z}_1 &= q^{-2} \mathsf{Z}_1 \mathsf{K}_0 \ , \qquad \mathsf{K}_0 \tilde{\mathsf{Z}}_1 = q^2 \tilde{\mathsf{Z}}_1 \mathsf{K}_0 \ , \\ \mathsf{K}_1 \mathsf{Z}_1 &= q^{2} \mathsf{Z}_1 \mathsf{K}_1 \ , \qquad \mathsf{K}_1 \tilde{\mathsf{Z}}_1 = q^{-2} \tilde{\mathsf{Z}}_1 \mathsf{K}_1 \ , \\ \big[\mathsf{Z}_1, \big[\mathsf{Z}_1, \big[\mathsf{Z}_1, \tilde{\mathsf{Z}}_1\big]_q\big]_{q^{-1}}\big] &= \rho_{diag} \mathsf{Z}_1 (\mathsf{K}_1 \mathsf{K}_1 - \mathsf{K}_0 \mathsf{K}_0) \mathsf{Z}_1, \\ \big[\tilde{\mathsf{Z}}_1, \big[\tilde{\mathsf{Z}}_1, \big[\tilde{\mathsf{Z}}_1, \mathsf{Z}_1\big]_q\big]_{q^{-1}}\big] &= \rho_{diag} \tilde{\mathsf{Z}}_1 (\mathsf{K}_0 \mathsf{K}_0 - \mathsf{K}_1 \mathsf{K}_1) \tilde{\mathsf{Z}}_1 \end{aligned} \tag{3.21}$$

with (2.19). The coaction map that is compatible with the coproduct of $U_q(\widehat{sl_2})$ here considered as well as the relations (3.21) is such that:

$$\begin{aligned} \delta(\mathsf{K}_0) &= q^{h_1} \otimes \mathsf{K}_0 \ , \qquad \delta(\mathsf{K}_1) = q^{h_0} \otimes \mathsf{K}_1 \ , \\ \delta(\mathsf{Z}_1) &= q^{h_0 + h_1} \otimes \mathsf{Z}_1 + (q^2 - q^{-2}) \big(q^{-1} e_0 q^{h_1} \otimes \mathsf{K}_0 + f_1 q^{h_0 + h_1} \otimes \mathsf{K}_1 \big) \ , \\ \delta(\tilde{\mathsf{Z}}_1) &= q^{h_0 + h_1} \otimes \tilde{\mathsf{Z}}_1 + (q^2 - q^{-2}) \big(f_0 q^{h_0 + h_1} \otimes \mathsf{K}_0 + q^{-1} e_1 q^{h_0} \otimes \mathsf{K}_1 \big) \ . \end{aligned} \tag{3.22}$$

For $N \rightarrow \infty$ it is straightforward to check that $\delta^{(N)}(\mathsf{K}_0)$, $\delta^{(N)}(\mathsf{K}_1)$, $\delta^{(N)}(\mathsf{Z}_1)$ and $\delta^{(N)}(\tilde{\mathsf{Z}}_1)$ act as (3.19) on $\mathcal{V}$, respectively.

Once again, another realization of the fundamental generators of $\mathcal{A}_q^{diag}$ can be proposed. As one can check, the elements

$$\begin{aligned}\overline{\mathsf{K}}_0 &= \bar{\epsilon}_- q^{-h_1}\ , \qquad \overline{\mathsf{K}}_1 = \bar{\epsilon}_+ q^{-h_0}\ ,\\ \overline{\mathsf{Z}}_1 &= (q^2-q^{-2})\big(\bar{\epsilon}_+ e_1 q^{-h_0-h_1} + \bar{\epsilon}_- q^{-1} f_0 q^{-h_1}\big)\ ,\\ \overline{\overline{\mathsf{Z}}}_1 &= (q^2-q^{-2})\big(\bar{\epsilon}_- e_0 q^{-h_0-h_1} + \bar{\epsilon}_+ q^{-1} f_1 q^{-h}\big)\end{aligned} \tag{3.23}$$

also satisfy the augmented $q-$Onsager algebra (3.21) and the appropriate coaction map is such that:

$$\begin{aligned}\bar{\delta}(\overline{\mathsf{K}}_0) &= \overline{\mathsf{K}}_0 \otimes q^{-h_1}\ , \qquad \bar{\delta}(\overline{\mathsf{K}}_1) = \overline{\mathsf{K}}_1 \otimes q^{-h_0}\ ,\\ \bar{\delta}(\overline{\mathsf{Z}}_1) &= \overline{\mathsf{Z}}_1 \otimes q^{-h_0-h_1} + (q^2-q^{-2})\big(\overline{\mathsf{K}}_1 \otimes e_1 q^{-h_0-h_1} + \overline{\mathsf{K}}_0 \otimes q^{-1} f_0 q^{-h_1}\big)\ ,\\ \bar{\delta}(\overline{\overline{\mathsf{Z}}}_1) &= \overline{\overline{\mathsf{Z}}}_1 \otimes q^{-h_0-h_1} + (q^2-q^{-2})\big(\overline{\mathsf{K}}_0 \otimes e_0 q^{-h_0-h_1} + \overline{\mathsf{K}}_1 \otimes q^{-1} f_1 q^{-h_0}\big)\ .\end{aligned} \tag{3.24}$$

Having identified $U_q(\widehat{sl_2})$ realizations of the fundamental generators of $\mathcal{A}_q$ and $\mathcal{A}_q^{diag}$, as well as left or right coaction maps which are compatible with the $U_q(\widehat{sl_2})$ coproduct (6.15), we can turn to the construction of infinite dimensional representations that will be useful in solving (3.1).

4. The current algebra $O_q(\widehat{sl_2})$ and $q-$vertex operators

The purpose of this Section is to construct infinite dimensional representations of the current algebra (3.3-3.8) that will find applications in the massive regime $-1<q<0$ of the XXZ open spin chain. Besides, we will show that the q-vertex operators of $U_q(\widehat{sl_2})$ are intertwiners of $\mathcal{A}_q$-modules or $\mathcal{A}_q^{diag}$-modules, giving an alternative support to the proposal of [JKKKM]. Let $\mathcal{V}_\zeta$ be the two-dimensional evaluation representation of $U_q(\widehat{sl_2})$ in the principal picture (see Appendix B) and consider first the realization (3.14). Following [DFJMN], type I and type II q-vertex operators can be introduced such that, respectively:

$$\begin{aligned}\chi(\zeta):&\qquad \mathcal{V}\rightarrow\mathcal{V}\otimes\mathcal{V}_\zeta\ ,\\ \overline{\chi}(\zeta):&\qquad \mathcal{V}\rightarrow\mathcal{V}_\zeta\otimes\mathcal{V}\ .\end{aligned}$$

According to the definition of the coaction map δ that is compatible with the realization (3.14), they satisfy (up to a scalar factor in the r.h.s):

$$\begin{aligned}\text{T}ype\ I:&\quad \chi(\zeta)\circ a = (id\times\pi_\zeta)\big[\delta(a)\big]\circ\chi(\zeta)\ ,\\ \text{T}ype\ II:&\quad \overline{\chi}(\zeta)\circ a = (\pi_\zeta\times id)\big[\delta(a)\big]\circ\overline{\chi}(\zeta)\qquad \forall a\in\mathcal{A}_q \text{ or } \mathcal{A}_q^{diag}\ .\end{aligned} \tag{4.1}$$

Writting q-vertex operators in the form[11]:

$$\begin{aligned}\chi(\zeta) &= \chi_+(\zeta)\otimes v_+ + \chi_-(\zeta)\otimes v_-\ ,\\ \overline{\chi}(\zeta) &= v_+\otimes\overline{\chi}_+(\zeta) + v_-\otimes\overline{\chi}_-(\zeta)\ ,\end{aligned}$$

two systems of equations follows from (4.1). Choosing $a\equiv \mathsf{W}_0,\mathsf{W}_1$ and using (3.16), the defining relations of type II q-vertex operators are given by:

$$\begin{aligned}\mathsf{W}_0\overline{\chi}_+(\zeta) &= q^{-1}\overline{\chi}_+(\zeta)\mathsf{W}_0 - k_+\zeta q^{-1}\overline{\chi}_-(\zeta)\ ,\\ \mathsf{W}_0\overline{\chi}_-(\zeta) &= q\overline{\chi}_-(\zeta)\mathsf{W}_0 - k_-\zeta^{-1}q\overline{\chi}_+(\zeta)\ ,\\ \mathsf{W}_1\overline{\chi}_+(\zeta) &= q\overline{\chi}_+(\zeta)\mathsf{W}_1 - k_+\zeta^{-1}q\overline{\chi}_-(\zeta)\ ,\\ \mathsf{W}_1\overline{\chi}_-(\zeta) &= q^{-1}\overline{\chi}_-(\zeta)\mathsf{W}_1 - k_-\zeta q^{-1}\overline{\chi}_+(\zeta)\ .\end{aligned} \tag{4.2}$$

[11] We set $v_+=\begin{pmatrix}1\\0\end{pmatrix}$ and $v_-=\begin{pmatrix}0\\1\end{pmatrix}$.

For type I q-vertex operators, the relations (4.1) hold for independent values of $k_\pm, \epsilon_\pm$. After simplications, the corresponding equations reduce to:

$$
\begin{array}{ll}
e_0\chi_+(\zeta) = \chi_+(\zeta)e_0\ , & e_1\chi_+(\zeta) + \zeta q^{h_1}\chi_-(\zeta) = \chi_+(\zeta)e_1\ ,\\
e_0\chi_-(\zeta) + \zeta q^{h_0}\chi_+(\zeta) = \chi_-(\zeta)e_0\ , & e_1\chi_-(\zeta) = \chi_-(\zeta)e_1\ ,\\
f_0q^{h_0}\chi_+(\zeta) + q\zeta^{-1}q^{h_0}\chi_-(\zeta) = \chi_+(\zeta)f_0q^{h_0}\ , & f_1q^{h_1}\chi_+(\zeta) = \chi_+(\zeta)f_1q^{h_1}\ ,\\
f_0q^{h_0}\chi_-(\zeta) = \chi_-(\zeta)f_0q^{h_0}\ , & f_1q^{h_1}\chi_-(\zeta) + q\zeta^{-1}q^{h_1}\chi_+(\zeta) = \chi_-(\zeta)f_1q^{h_1}\ ,\\
q^{h_0}\chi_\pm(\zeta) = q^{\pm1}\chi_\pm(\zeta)q^{h_0}\ , & q^{h_1}\chi_\pm(\zeta) = q^{\mp1}\chi_\pm(\zeta)q^{h_1}\ ,
\end{array}
\tag{4.3}
$$

which can be equally written, using the coproduct (6.15), as:

$$
\chi(\zeta)\circ x \quad = \quad (id\times\pi_\zeta)\big[\Delta(x)\big]\circ\chi(\zeta) \quad \text{for} \quad x\in\{e_i, f_iq^{h_i}, q^{h_i}\}\ . \tag{4.4}
$$

On the other hand, if we choose the second realization (3.17) of $\mathcal{A}_q$ instead, a similar analysis can be done using the coaction map $\overline{\delta}$. It leads to an alternative set of defining relations for type I and type II $q-$vertex operators.

Remark 3. *Type I $q-$vertex operators satisfy relations of the form (4.2) provided the substitutions:*

$$
\overline{\chi}_\pm(\zeta)\to\chi_\pm(\zeta^{-1})\ , \quad k_\pm\to\bar{k}_\pm\ , \quad \mathsf{W}_0\to\overline{\mathsf{W}}_1\ , \quad \mathsf{W}_1\to\overline{\mathsf{W}}_0\ . \tag{4.5}
$$

Type II $q-$vertex operators can be defined similarly. For generic values of $\bar{k}_\pm, \bar{\epsilon}_\pm$ the defining relations simplify to:

$$
\overline{\chi}(\zeta)\circ x \quad = \quad (\pi_\zeta\times id)\big[\Delta(x)\big]\circ\overline{\chi}(\zeta) \quad \textit{for} \quad x\in\{e_iq^{-h_i}, f_i, q^{-h_i}\}\ . \tag{4.6}
$$

The same analysis applies to the fundamental generators of $\mathcal{A}_q^{diag}$. Using the realization (3.20) with coaction map δ, two sets of equations are obtained. For instance, according to the coaction map (3.22) type II $q-$vertex operators are defined by:

$$
\begin{array}{rcl}
\mathsf{K}_0\overline{\chi}_\pm(\zeta) &=& q^{\mp1}\overline{\chi}_\pm(\zeta)\mathsf{K}_0\ ,\\
\mathsf{K}_1\overline{\chi}_\pm(\zeta) &=& q^{\pm1}\overline{\chi}_\pm(\zeta)\mathsf{K}_1\ ,\\
\mathsf{Z}_1\overline{\chi}_+(\zeta) &=& \overline{\chi}_+(\zeta)\mathsf{Z}_1\ ,\\
\mathsf{Z}_1\overline{\chi}_-(\zeta) &=& \overline{\chi}_-(\zeta)\mathsf{Z}_1 - (q^2-q^{-2})\big(\zeta q^{-1}\overline{\chi}_+(\zeta)\mathsf{K}_0 + \zeta^{-1}q\overline{\chi}_+(\zeta)\mathsf{K}_1\big)\ ,\\
\tilde{\mathsf{Z}}_1\overline{\chi}_+(\zeta) &=& \overline{\chi}_+(\zeta)\tilde{\mathsf{Z}}_1 - (q^2-q^{-2})\big(\zeta q^{-1}\overline{\chi}_-(\zeta)\mathsf{K}_1 + \zeta^{-1}q\overline{\chi}_-(\zeta)\mathsf{K}_0\big)\ ,\\
\tilde{\mathsf{Z}}_1\overline{\chi}_-(\zeta) &=& \overline{\chi}_-(\zeta)\tilde{\mathsf{Z}}_1\ .
\end{array}
\tag{4.7}
$$

The defining relations of type I $q-$vertex operators reduce to (4.4).

Remark 4. *Type I $q-$vertex operators satisfy relations of the form (4.7) provided the substitutions:*

$$
\overline{\chi}_\pm(\zeta)\to\chi_\pm(\zeta^{-1})\ , \quad \mathsf{K}_0\to\overline{\mathsf{K}}_1\ , \quad \mathsf{K}_1\to\overline{\mathsf{K}}_0\ , \quad \mathsf{Z}_1\to\overline{\tilde{\mathsf{Z}}}_1\ , \quad \tilde{\mathsf{Z}}_1\to\overline{\mathsf{Z}}_1\ . \tag{4.8}
$$

Type II $q-$vertex operators can be defined similarly, leading to (4.6)

More generally, for a given realization and corresponding coaction map the defining relations generalizing (4.2) or (4.7) or alternatively those associated with (4.5) or (4.8) - satisfied by the intertwiners for *any* element of $\mathcal{A}_q$ or $\mathcal{A}_q^{diag}$, respectively - can be obtained using the properties of the current algebra $O_q(\widehat{sl_2})$. Two different types of coaction map that generalize either δ or $\overline{\delta}$ have to be considered to this end. Namely, with minor changes in the results[12] of [[BSh],Proposition 2.2], a left coaction map $\delta':\mathcal{A}_q\to U_q(sl_2)\otimes\mathcal{A}_q$ which preserves all defining relations (3.3)-(3.8) follows. By straightforward calculations, the relations generalizing (4.2) or (4.7) according to (3.9) and (3.10) follow from (4.1). Combining these, we eventually find that the $q-$vertex operators must satisfy:

[12]Starting from a solution of the reflection equation $K(\zeta)$ in terms of the currents, it is known that $R_{12}(\zeta/v)K_1(\zeta)R_{12}(\zeta v)$ is also a solution of the reflection equation. This property was used in [BSh] to define a coaction map δ'.

$$
\begin{aligned}
(4.9)\quad \mathcal{W}_-(\zeta)\overline{\chi}_-(v) &= \frac{\kappa(v\,\zeta)\kappa(-v\,\zeta^{-1}q^{-1})}{U(\zeta)-U(v^{-1}q)}\Big((q^{-1}U(\zeta)-U(v^{-1}\sqrt{q}))\overline{\chi}_-(v)\mathcal{W}_-(\zeta)+q\frac{q-q^{-1}}{q+q^{-1}}\overline{\chi}_-(v)\mathcal{W}_+(\zeta)\\
&\qquad -v\,q^{-1}\frac{(q-q^{-1})}{(q+q^{-1})^2}\overline{\chi}_+(v)\mathcal{Z}_-(\zeta)\Big)\\
\mathcal{W}_+(\zeta)\overline{\chi}_-(v) &= \frac{\kappa(v\,\zeta)\kappa(-v\,\zeta^{-1}q^{-1})}{U(\zeta)-U(v^{-1}q)}\Big((q\,U(\zeta)-U(\sqrt{q}\,v^{-1}))\overline{\chi}_-(v)\mathcal{W}_+(\zeta)-q^{-1}\frac{q-q^{-1}}{q+q^{-1}}\overline{\chi}_-(v)\mathcal{W}_-(\zeta)\\
&\qquad -v^{-1}q\,\frac{(q-q^{-1})}{(q+q^{-1})^2}\overline{\chi}_+(v)\mathcal{Z}_-(\zeta)\Big)\\
\mathcal{Z}_-(\zeta)\overline{\chi}_-(v) &= \kappa(v\,\zeta)\kappa(-v\,\zeta^{-1}q^{-1})\Big(\overline{\chi}_-(v)\mathcal{Z}_-(\zeta)\Big)\\
\mathcal{Z}_+(\zeta)\overline{\chi}_-(v) &= \frac{\kappa(v\,\zeta)\kappa(-v\,\zeta^{-1}q^{-1})}{U(\zeta)-U(v^{-1}q)}\Big((U(\zeta)-U(v\,q^{-1}))\overline{\chi}_-(v)\mathcal{Z}_+(\zeta)\\
&\qquad -(q^2-q^{-2})(v\,q^{-1}\,U(\zeta)-v^{-1}q)\overline{\chi}_+(v)\mathcal{W}_-(\zeta)\\
&\qquad -(q^2-q^{-2})(v^{-1}q\,U(\zeta)-v\,q^{-1})\overline{\chi}_+(v)\mathcal{W}_+(\zeta)\Big)\ .
\end{aligned}
$$

Changing $\mathcal{W}_\pm(\zeta)\to\mathcal{W}_\mp(\zeta)$, $\mathcal{Z}_\pm(\zeta)\to\mathcal{Z}_\mp(\zeta)$,$\overline{\chi}_\pm\to\overline{\chi}_\mp$ in above formula, the action of each current on $\overline{\chi}_+(v)$ follows. Note that the prefactor in the r.h.s of (4.9) comes from the definition of the $R-$matrix (6.17) with (6.19), which automatically appears in the explicit form of the coaction. As one can check, expanding the currents according to (3.9) or (3.10) the defining relations (4.2) or (4.7), respectively, are exactly reproduced at the leading order[13] in $U(\zeta)$.

Remark 5. *A right coaction map $\overline{\delta}'$: $\mathcal{A}_q\to\mathcal{A}_q\otimes U_q(sl_2)$ which preserves all defining relations (3.3)-(3.8) can be considered instead. Type I $q-$vertex operators are defined accordingly, in which case the defining relations take the form (4.9) provided the substitutions:*

$$
(4.10)\qquad \overline{\chi}_\pm(\zeta)\to\chi_\pm(\zeta^{-1})\ ,\quad \mathcal{W}_\pm(\zeta)\to\overline{\mathcal{W}}_\mp(\zeta)\ ,\quad \mathcal{Z}_\pm(\zeta)\to\overline{\mathcal{Z}}_\mp(\zeta)\ .
$$

Note that at the leading order in $U(\zeta)$, one recovers the defining relations (4.5), (4.8).

The identification of the $q-$vertex operators associated with $\mathcal{A}_q$ and $\mathcal{A}_q^{diag}$ is now straightforward. On one hand, we have observed that $\mathcal{A}_q$ and $\mathcal{A}_q^{diag}$ can be both interpreted as a left (resp. right) coideal subalgebra of $U_q(\widehat{sl_2})$ using the realizations (3.14) or (3.20) (resp. (3.17) or (3.23)). According to the choice of realization and corresponding coaction map, we have also identified the relations satisfied by type I and type II $q-$vertex operators, namely (4.2), (4.4), (4.5), (4.6), (4.7), (4.8) and, more generally, the relations (4.9) and (4.10). In particular, the relations (4.4), (4.6) are nothing but the defining relations of type I and type II q-vertex operators of $U_q(\widehat{sl_2})$. Then, let $V(\Lambda_i)$, $i=0,1$, denote the integrable highest weight level one[14] modules of $U_q(\widehat{sl_2})$. Recall that type I and type II q-vertex operators act as:

$$
\begin{aligned}
\text{Type } I: &\quad \Phi^{(1-i,i)}(\zeta): \quad V(\Lambda_i)\to V(\Lambda_{1-i})\otimes\mathcal{V}_\zeta\ ,\\
\text{Type } II: &\quad \Psi^{*(1-i,i)}(\zeta): \quad V(\Lambda_i)\to \mathcal{V}_\zeta\otimes V(\Lambda_{1-i})
\end{aligned}
$$

and the q-vertex operators can be written in the form [DFJMN]

$$
\begin{aligned}
\Phi^{(1-i,i)}(\zeta) &= \Phi_+^{(1-i,i)}(\zeta)\otimes v_+ + \Phi_-^{(1-i,i)}(\zeta)\otimes v_-\ ,\\
\Psi^{*(1-i,i)}(\zeta) &= v_+\otimes\Psi_+^{*\,(1-i,i)}(\zeta)+v_-\otimes\Psi_-^{*\,(1-i,i)}(\zeta)\ .
\end{aligned}
$$

Using the explicit realizations (3.14) or (3.20) or, alternatively (3.17) or (3.23), it is straightforward to check that the following maps

$$
(4.11)\qquad \chi(\zeta)\to\Phi^{(1-i,i)}(\zeta)\qquad \overline{\chi}(\zeta)\to\Psi^{*(1-i,i)}(\zeta)\qquad \text{for any}\quad i=0,1
$$

[13] Note that $\kappa(v\,\zeta)\kappa(-v\,\zeta^{-1}q^{-1})$ is invariant under the substition $\zeta\to-\zeta^{-1}q^{-1}$.

[14] At level one, note that $q^{h_0+h_1}=q$.

provide explicit realizations of type I and type II q-vertex operators of $\mathcal{A}_q$ and $\mathcal{A}_q^{diag}$. The proof solely uses the defining relations of type I and type II $q-$vertex operators of $U_q(\widehat{sl_2})$.

Now, an explicit expression for the $O_q(\widehat{sl_2})$ currents such that all relations (4.9) or (4.10) are satisfied is required for completeness. To this end, recall that the currents admit certain realizations in terms of elements satisfying a Zamolodchikov-Faddeev algebra: adapting the results of [[BB2], Proposition 3.2] one shows that all defining relations (3.3)-(3.8) are satisfied by (4.11):

$$\begin{aligned} \mathcal{W}_\pm(\zeta) &\rightarrow \frac{\zeta q {\Psi^*}_\pm^{(1-i,i)}(\zeta^{-1}){\Psi^*}_\mp^{(1-i,i)}(-\zeta\, q) + \zeta^{-1}q^{-1}\, {\Psi^*}_\mp^{(1-i,i)}(\zeta^{-1}){\Psi^*}_\pm^{(1-i,i)}(-\zeta\, q)}{\zeta^2q^2-\zeta^{-2}q^{-2}} , \\ \mathcal{Z}_\pm(\zeta) &\rightarrow (q+q^{-1}){\Psi^*}_\pm^{(1-i,i)}(\zeta^{-1}){\Psi^*}_\pm^{(1-i,i)}(-\zeta\, q) . \end{aligned} \tag{4.12}$$

Indeed, type I and type II $q-$vertex operators of $U_q(\widehat{sl_2})$ satisfy the Zamolodchikov-Faddeev algebras (6.20), (6.21), respectively (see [BB2] for details). Then, by straightforward calculations one checks that (4.12) exactly reproduces (4.9) using (4.11). A similar conclusion follows by substituting $\mathcal{W}_\pm(\zeta)\rightarrow\overline{\mathcal{W}}_\mp(\zeta)$, $\mathcal{Z}_\pm(\zeta)\rightarrow\overline{\mathcal{Z}}_\mp(\zeta)$ in (4.12) and using (4.10). All these results confirm the proposal (4.11).

5. Eigenstates of $O_q(\widehat{sl_2})$ currents and diagonalization

In [Bas3], two finite dimensional eigenbasis of $\mathcal{A}_q$ were explicitely constructed, which unravealed a generalization of the property of tridiagonal pairs [Ter2] for all generators of $\mathcal{A}_q$: as described in [BK3], in a certain basis of the truncated finite vector space $\mathcal{V}^{(N)}$ any generator of $\mathcal{A}_q$ act as a block tridiagonal matrix. Clearly, this property extends to the generators of $\mathcal{A}_q^{diag}$. Whether this property generalizes to infinite dimensional representations is an interesting question which goes beyond the scope of this article. However, based on the conjecture that the level one irreducible highest weight $U_q(\widehat{sl_2})$ representation indexed 'i' is embedded into the half-infinite vector space '$\mathcal{V}$' using $q-$vertex operators [JKKKM], for $-1<q<0$ it is possible to construct two eigenbasis of $O_q(\widehat{sl_2})$ currents. Using previous results, for the discussion below we focus on the spectral problem associated with the currents:

$$\overline{\mathcal{W}}_\pm^{(i)}(\zeta)\ ,\ \overline{\mathcal{Z}}_\pm^{(i)}(\zeta):\quad V(\Lambda_i)\rightarrow V(\Lambda_i)\qquad \text{for}\qquad i=0,1$$

with

$$\overline{\mathcal{W}}_\pm^{(i)}(\zeta) = \frac{\zeta q \Phi_\mp^{(i,1-i)}(\zeta)\Phi_\pm^{(1-i,i)}(-\zeta^{-1}q^{-1}) + \zeta^{-1}q^{-1}\,\Phi_\pm^{(i,1-i)}(\zeta)\Phi_\mp^{(1-i,i)}(-\zeta^{-1}q^{-1})}{\zeta^2q^2-\zeta^{-2}q^{-2}} , \tag{5.1}$$

$$\overline{\mathcal{Z}}_\pm^{(i)}(\zeta) = (q+q^{-1})\Phi_\mp^{(i,1-i)}(\zeta)\Phi_\mp^{(1-i,i)}(-\zeta^{-1}q^{-1}) . \tag{5.2}$$

On one hand, consider the spectral problem

$$\overline{\mathcal{W}}_\pm^{(i)}(-\zeta^{-1}q^{-1})|B_\pm\rangle = \lambda_\pm(-\zeta^{-1}q^{-1})|B_\pm\rangle \qquad \text{for}\qquad i=0,1 . \tag{5.3}$$

Acting with $g\Phi_\mp(\zeta^{-1})$ (or, alternatively $g\Phi_\pm(\zeta^{-1})$) from the left on this equation and using the properties of $q-$vertex operators (see Appendix C), it yields to:

$$\frac{\zeta^{\pm1}}{\zeta^2-\zeta^{-2}}\Phi_\mp^{(1-i,i)}(\zeta)|B_+\rangle = g\lambda_+(-\zeta^{-1}q^{-1})\Phi_\mp^{(1-i,i)}(\zeta^{-1})|B_+\rangle , \tag{5.4}$$

$$\frac{\zeta^{\mp1}}{\zeta^2-\zeta^{-2}}\Phi_\mp^{(1-i,i)}(\zeta)|B_-\rangle = g\lambda_-(-\zeta^{-1}q^{-1})\Phi_\mp^{(1-i,i)}(\zeta^{-1})|B_-\rangle . \tag{5.5}$$

For each current, by straightforward calculations one finds the 'minimal' solution:

$$|B_+\rangle = e^{\tilde{F}_0}|0\rangle \qquad \text{and} \qquad |B_-\rangle = e^{\alpha/2}e^{\tilde{F}_0}|0\rangle \tag{5.6}$$

where

$$\tilde{F}_0 = -\frac{1}{2}\sum_{n=1}^\infty \frac{nq^{6n}}{[2n][n]}a_{-n}^2 - \sum_{n=1}^\infty \frac{q^{5n/2}(1-q^n)\theta_n}{[2n]}a_{-n} \quad \text{with} \quad \theta_n=1(0)\ \text{ for }\ n \text{ even (odd)} .$$

Accordingly, the spectrum reads:

$$\lambda_+(\zeta) = \lambda_-(\zeta) = \frac{1}{g}\frac{\zeta^{-1}q^{-1}}{\zeta^2q^2 - \zeta^{-2}q^{-2}}\frac{\delta(\zeta^2q^2)}{\delta(\zeta^{-2}q^{-2})} \quad \text{where} \quad \delta(z) = \frac{(q^6z^2;q^8)_\infty}{(q^8z^2;q^8)_\infty} \ . \tag{5.7}$$

Remark 6. *The eigenvector $|B_+\rangle$ (resp. $|B_-\rangle$) coincides exactly with $|0\rangle_B|_{r\to 0}$ (resp. $e^{\alpha/2}|1\rangle_B|_{r\to\infty}$) in* [JKKKMW]. *Also, the eigenvectors $|B_\pm\rangle$ do not depend on the index $i = 0,1$. As a consequence, $\overline{\mathcal{W}}_\pm^{(i)}(\zeta)|B_\pm\rangle = \overline{\mathcal{W}}_\pm^{(1-i)}(\zeta)|B_\pm\rangle$.*

More generally, two families of eigenstates of $\overline{\mathcal{W}}_\pm^{(i)}(\zeta)$ can be constructed using the properties of type II $q-$vertex operators: starting from $|B_\pm\rangle$, for any $i = 0,1$ the commutation relations (6.22) imply:

$$\overline{\mathcal{W}}_\pm^{(i)}(-\zeta^{-1}q^{-1})\Psi^*_{\mu_1}(\xi_1)...\Psi^*_{\mu_m}(\xi_m)|B_\pm\rangle = \lambda_\pm(-\zeta^{-1}q^{-1};\xi_1,...,\xi_m)\Psi^*_{\mu_1}(\xi_1)...\Psi^*_{\mu_m}(\xi_m)|B_\pm\rangle \tag{5.8}$$

with

$$\lambda_\pm(\zeta;\xi_1,...,\xi_m) = \prod_{j=1}^m \tau(\zeta/\xi_j)\tau(\zeta\xi_j)\lambda_\pm(\zeta) \ .$$

The action of other currents for any $i = 0,1$ immediately follows:

$$\begin{aligned}\overline{\mathcal{Z}}_\epsilon^{(i)}(-\zeta^{-1}q^{-1})\Psi^*_{\mu_1}(\xi_1)...\Psi^*_{\mu_m}(\xi_m)|B_{\epsilon'}\rangle \quad &= \\ (q+q^{-1})\zeta^{1-\epsilon\epsilon'}\frac{\delta(\zeta^2q^2)}{\delta(\zeta^{-2}q^{-2})}\ \prod_{j=1}^m \tau(\zeta/\xi_j)\tau(\zeta\xi_j)\ \ \Phi_\epsilon^{*(i,1-i)}(\zeta^{-1})\Phi_{-\epsilon}^{(1-i,i)}(\zeta^{-1})\Psi^*_{\mu_1}(\xi_1)...\Psi^*_{\mu_m}(\xi_m)|B_{\epsilon'}\rangle \ .\end{aligned}$$

Having explicit expressions for $\overline{\mathcal{W}}_\pm^{(i)}(\zeta)$ currents' eigenstates, following [BK3] we now turn to the diagonalization of the Hamiltonian (3.1) in the massive regime $-1 < q < 0$ within Onsager's approach. In Section 2, we have introduced the corresponding transfer matrix in terms of the generating function of all mutually commuting quantities (3.11) that form the so-called $q-$Dolan-Grady hierarchy, by analogy with the finite size case. According to (3.3)-(3.8), the commutation and invertibility relations of the $q-$vertex operators (see Appendix C) and (5.1), (5.2), observe that the following relations are satisfied:

$$\big[\overline{\mathcal{I}}^{(i)}(\zeta),\overline{\mathcal{I}}^{(i)}(\xi)\big] = 0 \ , \quad \overline{\mathcal{I}}^{(i)}(\zeta) = \kappa(-q\zeta^2)\overline{\mathcal{I}}^{(i)}(-\zeta^{-1}q^{-1}) \ , \quad g(\zeta^2-\zeta^{-2})\overline{\mathcal{I}}^{(i)}(\zeta)|_{\zeta=1} = \epsilon_+ + \epsilon_- \ ,$$
$$-g^2(\zeta^2-\zeta^{-2})^2\overline{\mathcal{I}}^{(i)}(\zeta)\overline{\mathcal{I}}^{(i)}(\zeta^{-1}) = (\epsilon_++\epsilon_-)^2 + (\zeta-\zeta^{-1})^2\epsilon_+\epsilon_- - \frac{(\zeta^2-\zeta^{-2})^2}{(q-q^{-1})^2}k_+k_- \ .$$

As a consequence, the following contraints on the normalization factor in terms of the boundary parameters follow:

$$\begin{aligned}\frac{\rho(\zeta)}{\rho(-q^{-1}\zeta^{-1})} \quad &= \quad -\frac{1}{\kappa(-q\zeta^2)}\frac{(\zeta^2-\zeta^{-2})}{(q^2\zeta^2-q^{-2}\zeta^{-2})} \ , \\ \rho(\zeta)\rho(\zeta^{-1}) \quad &= \quad (\epsilon_++\epsilon_-)^2 + (\zeta-\zeta^{-1})^2\epsilon_+\epsilon_- - \frac{(\zeta^2-\zeta^{-2})^2}{(q-q^{-1})^2}k_+k_- \ , \\ \rho(1) \quad &= \quad \epsilon_++\epsilon_- \ .\end{aligned} \tag{5.9}$$

Note that if one writes (2.7) solely in terms of $q-$vertex operators, one recovers exactly the transfer matrix proposed in [[JKKKMW], eq. (2.13)]. In this case, the scalar solution (2.4) of the reflection equation associated with the right boundary can be explicitely exhibited.

We are now in position to study the spectral problem associated with (3.11) for any choice of boundary parameters $\epsilon_\pm, k_\pm$. In the present article, for simplicity we will focus on two special cases: first, we will study the case of *diagonal* boundary conditions $\epsilon_\pm \neq 0, k_\pm = 0$: it has been considered in details in the literature [JKKKM] and will serve as a check of the approach here presented. Then, we will consider the case of upper ($\epsilon_\pm \neq 0,\ k_+ \neq 0,\ k_- = 0$) or lower ($\epsilon_\pm \neq 0,\ k_- \neq 0,\ k_+ = 0$) non-diagonal boundary conditions: the results here obtained can be compared with the known ones obtained within the Bethe ansatz approach [BCRS]. Note that the explicit expression for the vacuum vectors for upper or lower non-diagonal boundary conditions allows to derive an integral representation for correlation fonctions, following [DFJMN, JKKKM]. This will be considered elsewhere.

5.1. **The diagonal case revisited.** For *diagonal* boundary conditions $k_\pm = 0$ and $\epsilon_\pm \neq 0$, define $v^2 \equiv r = -\epsilon_+/\epsilon_-$. By straightforward calculations, the solution $\rho(\zeta)$ to the constraints (5.9) that is compatible with the action of the $q-$vertex operators[15] is given by:

$$\rho(\zeta) = (\zeta\epsilon_- + \zeta^{-1}\epsilon_+)\frac{\delta(\zeta^{-2})}{\delta(\zeta^2)}\frac{\tilde{\varphi}(\zeta^{-2};r)}{\tilde{\varphi}(\zeta^2;r)} \tag{5.10}$$

where

$$\tilde{\varphi}(z;r) = \frac{(q^4rz;q^4)_\infty}{(q^2rz;q^4)_\infty} \quad \text{and} \quad \delta(z) = \frac{(q^6z^2;q^8)_\infty}{(q^8z^2;q^8)_\infty} \ .$$

Now, starting from the eigenstates $|B_\pm\rangle$ of $\overline{\mathcal{W}}_\pm^{(i)}(\zeta)$ defined by (5.6), we are looking for the vacuum vectors of the transfer matrix (3.12) for $k_\pm = 0$.

First vacuum vector $|0\rangle_B$: Define

$$|0\rangle_B \equiv e^{f(v)}|B_+\rangle \quad \text{where} \quad f(v) = -\sum_{n=1}^{\infty}\frac{a_{-n}}{[2n]}q^{7n/2}v^{2n} \ .$$

The action of the $q-$vertex operators on the exponential term is such that:

$$\begin{aligned}
\Phi_-^{(1-i,i)}(\zeta)e^{f(v)} &= \tilde{\varphi}(\zeta^{-2};v^2)e^{f(v)}\Phi_-^{(1-i,i)}(\zeta) \ , \\
\Phi_+^{(1-i,i)}(\zeta)e^{f(v)} &= \tilde{\varphi}(\zeta^{-2};v^2)\Big((1-v^2\zeta^{-2})\ e^{f(v)}\Phi_+^{(1-i,i)}(\zeta) + v^2(1-q^2)\zeta^{-1}e^{f(v)}\Phi_-^{(1-i,i)}(\zeta)x_{-1}^-\Big) \ ,
\end{aligned}$$

where we introduced the Drinfeld's generator

$$x_{-1}^- = \oint_{C_1}\frac{dw}{2\pi i}X^-(w) \ .$$

Using (5.4), (5.5) and noticing that $x_{-1}^-|B_+\rangle = 0$ by straightforward calculations, one derives

$$\Phi_-^{(1,0)}(\zeta)\ e^{f(v)}|B_+\rangle = \frac{\delta(\zeta^{-2})\tilde{\varphi}(\zeta^{-2};v^2)}{\delta(\zeta^2)\tilde{\varphi}(\zeta^2;v^2)}\ \Phi_-^{(1,0)}(\zeta^{-1})\ e^{f(v)}|B_+\rangle \ , \tag{5.11}$$

$$\Phi_+^{(1,0)}(\zeta)\ e^{f(v)}|B_+\rangle = \frac{(\zeta^2-v^2)}{(1-v^2\zeta^2)}\frac{\delta(\zeta^{-2})\tilde{\varphi}(\zeta^{-2};v^2)}{\delta(\zeta^2)\tilde{\varphi}(\zeta^2;v^2)}\ \Phi_+^{(1,0)}(\zeta^{-1})\ e^{f(v)}|B_+\rangle \ . \tag{5.12}$$

Note that this result is in agreement with [JKKKMW], although the notations here differ. As a consequence, the action of the currents on the state $|0\rangle_B$ is given by:

$$\begin{aligned}
\overline{\mathcal{W}}_+^{(0)}(-q^{-1}\zeta^{-1})|0\rangle_B &= \frac{\varphi(\zeta^{-2};r)}{\varphi(\zeta^2;r)}\left(\frac{1}{g}\frac{(\zeta-r\zeta^{-1})}{(\zeta^2-\zeta^{-2})(1-r\zeta^2)} - \frac{r\zeta}{1-r\zeta^2}\Phi_-^{*\ (0,1)}(\zeta^{-1})\Phi_-^{(1,0)}(\zeta^{-1})\right)|0\rangle_B \ , \\
\overline{\mathcal{W}}_-^{(0)}(-q^{-1}\zeta^{-1})|0\rangle_B &= \frac{\varphi(\zeta^{-2};r)}{\varphi(\zeta^2;r)}\left(\frac{1}{g}\frac{\zeta^2(\zeta-r\zeta^{-1})}{(\zeta^2-\zeta^{-2})(1-r\zeta^2)} - \frac{\zeta}{1-r\zeta^2}\Phi_-^{*\ (0,1)}(\zeta^{-1})\Phi_-^{(1,0)}(\zeta^{-1})\right)|0\rangle_B
\end{aligned}$$

where the notation $\varphi(z;r) = \delta(z)\tilde{\varphi}(z;r)$ has been introduced to fit with [JKKKMW]. Combining both expressions together according to (3.11), the off-diagonal contribution cancels. Using (5.10), in agreement with [JKKKMW] one finds:

$$t^{(0)}(\zeta)|_{k_\pm=0}|0\rangle_B = 1\ |0\rangle_B \ . \tag{5.13}$$

Second vacuum vector $|1\rangle_B$: Similar analysis can be done for the second eigenstate, denoted $|1\rangle_B$ in [JKKKMW]. Define

$$|1\rangle_B \equiv e^{-f(-q^{-1}v^{-1})}|B_-\rangle \ .$$

[15] Indeed, another solution to (5.9) may be considered.

By straightforward calculations, one finds that

$$\Phi_-^{(0,1)}(\zeta)\ e^{-f(-q^{-1}v^{-1})}|B_-\rangle = \Lambda(\zeta;v^2)\frac{\delta(\zeta^{-2})\tilde{\varphi}(\zeta^{-2};v^2)}{\delta(\zeta^2)\tilde{\varphi}(\zeta^2;v^2)}\ \Phi_-^{(0,1)}(\zeta^{-1})\ e^{-f(-q^{-1}v^{-1})}|B_-\rangle\ , \tag{5.14}$$

$$\Phi_+^{(0,1)}(\zeta)\ e^{-f(-q^{-1}v^{-1})}|B_-\rangle = \Lambda(\zeta;v^2)\frac{(\zeta^2-v^2)}{(1-v^2\zeta^2)}\frac{\delta(\zeta^{-2})\tilde{\varphi}(\zeta^{-2};v^2)}{\delta(\zeta^2)\tilde{\varphi}(\zeta^2;v^2)}\ \Phi_+^{(0,1)}(\zeta^{-1})\ e^{-f(-q^{-1}v^{-1})}|B_-\rangle\ . \tag{5.15}$$

where

$$\Lambda(\zeta;v^2) = \zeta^2\frac{\tilde{\varphi}(q^{-2}\zeta^2;v^{-2})\tilde{\varphi}(\zeta^2;v^2)}{\tilde{\varphi}(q^{-2}\zeta^{-2};v^{-2})\tilde{\varphi}(\zeta^{-2};v^2)}\ .$$

The action of the currents $\overline{\mathcal{W}}_\pm^{(1)}(-q^{-1}\zeta^{-1})$ on $|1\rangle_B$ follows, which leads to

$$t^{(1)}(\zeta)|_{k_\pm=0}|1\rangle_B = \Lambda(\zeta;r)\ |1\rangle_B\ .$$

in agreement with [JKKKMW]. Finally, according to the observation that (see Appendix C and (4.12)):

$$\overline{\mathcal{W}}_\pm^{(i)}(\zeta)\Psi_\mu^{*(i,1-i)}(\xi) = \tau(\zeta/\xi)\tau(\zeta\xi)\ \Psi_\mu^{*(i,1-i)}(\xi)\overline{\mathcal{W}}_\pm^{(i)}(\zeta)\ ,$$

more general eigenstates of the transfer matrix are generated using type II $q-$vertex operators. In agreement with [JKKKM], it follows:

$$t^{(i)}(\zeta)|_{k_\pm=0}\ \Psi_{\mu_1}^*(\xi_1)...\Psi_{\mu_m}^*(\xi_m)|i\rangle_B = \Lambda^{(i)}(\zeta;r)\prod_{j=1}^m\tau(\zeta/\xi_j)\tau(\zeta\xi_j)\ \Psi_{\mu_1}^*(\xi_1)...\Psi_{\mu_m}^*(\xi_m)|i\rangle_B\ .$$

where $\Lambda^{(0)}(\zeta;r)=1$ and $\Lambda^{(1)}(\zeta;r)=\Lambda(\zeta;r)$. From (3.12), the energy levels are derived and expressed in terms of Jacobi elliptic functions [JKKKM].

5.2. **Upper or lower non-diagonal boundary conditions.** Let us consider the Hamiltonian (3.1) with upper *non-diagonal* boundary conditions $\epsilon_\pm\neq 0, k_+\neq 0$ and $k_-=0$, keeping above parametrization $v^2\equiv r=-\epsilon_+/\epsilon_-$. Clearly, in this case the solution $\rho(\zeta)$ is also given by (5.10) as the product of non-diagonal parameters k_+k_- vanishes. According to the results of Section 3, let us consider by analogy with (3.17) the following operators:

$$\overline{\mathsf{w}}_0 \quad = \quad k'_+q^{-1}e_1q^{-h_1}+k'_-f_1 \qquad \text{and} \qquad \overline{\mathsf{w}}_1 = k''_-q^{-1}e_0q^{-h_0}+k''_+f_0\ , \tag{5.16}$$

where the parameters $k'_\pm, k''_\pm$ are not determined yet. Their action on type I $q-$vertex operators are deduced from (4.3). Now, define:

$$\overline{\mathsf{w}}_0^{(\pm)}\equiv\overline{\mathsf{w}}_0|_{k'_\mp=0} \qquad \text{and} \qquad \overline{\mathsf{w}}_1^{(\pm)}\equiv\overline{\mathsf{w}}_1|_{k''_\mp=0}\ .$$

By straightforward calculation, it follows:

$$\begin{aligned}
\Phi_-(\zeta)(\overline{\mathsf{w}}_0^{(+)})^n &= q^n(\overline{\mathsf{w}}_0^{(+)})^n\Phi_-(\zeta)\ ,\\
\Phi_+(\zeta)(\overline{\mathsf{w}}_0^{(+)})^n &= q^{-n}(\overline{\mathsf{w}}_0^{(+)})^n\Phi_+(\zeta)+k'_+\zeta[n]_q(\overline{\mathsf{w}}_0^{(+)})^{n-1}\Phi_-(\zeta)\ ,\\
\Phi_-(\zeta)(\overline{\mathsf{w}}_0^{(-)})^n &= q^n(\overline{\mathsf{w}}_0^{(-)})^n\Phi_-(\zeta)+k'_-\zeta^{-1}[n]_q(\overline{\mathsf{w}}_0^{(-)})^{n-1}\Phi_+(\zeta)\ ,\\
\Phi_+(\zeta)(\overline{\mathsf{w}}_0^{(-)})^n &= q^{-n}(\overline{\mathsf{w}}_0^{(-)})^n\Phi_+(\zeta)
\end{aligned}$$

and similarly for $\overline{\mathsf{w}}_1^{(\pm)}$, provided the substitutions $\overline{\mathsf{w}}_0^{(\pm)}\to\overline{\mathsf{w}}_1^{(\pm)}$, $k'_\pm\to k''_\pm$ and $\zeta\to\zeta^{-1}$ in above commutation relations. According to (5.11), (5.12), (5.14), (5.15), let us consider the following combinations:

$$|+;0\rangle = \sum_{n=0}^\infty\frac{q^{n(n-1)/2}}{[n]_q!}(\overline{\mathsf{w}}_1^{(+)})^n|0\rangle_B \qquad \text{and} \qquad |+;1\rangle = \sum_{n=0}^\infty\frac{q^{-n(n-1)/2}}{[n]_q!}(\overline{\mathsf{w}}_0^{(+)})^n|1\rangle_B\ . \tag{5.17}$$

Acting with type I $q-$vertex operators, it is easy to show that:

$$\Phi_-^{(1,0)}(\zeta)\ |+;0\rangle = \frac{\varphi(\zeta^{-2};r)}{\varphi(\zeta^2;r)}\ \Phi_-^{(1,0)}(\zeta^{-1})\ |+;0\rangle\ , \tag{5.18}$$

$$\Phi_+^{(1,0)}(\zeta)\ |+;0\rangle = \frac{\varphi(\zeta^{-2};r)}{\varphi(\zeta^2;r)}\ \left(\frac{(\zeta^2-v^2)}{(1-v^2\zeta^2)}\Phi_+^{(1,0)}(\zeta^{-1})\ - k''_+\frac{\zeta(\zeta^2-\zeta^{-2})}{1-v^2\zeta^2}\ \Phi_-^{(1,0)}(\zeta^{-1})\ \right)\ |+;0\rangle \tag{5.19}$$

and

$$\Phi_-^{(0,1)}(\zeta)\ |+;1\rangle = \Lambda(\zeta;v^2)\frac{\varphi(\zeta^{-2};r)}{\varphi(\zeta^2;r)}\ \Phi_-^{(0,1)}(\zeta^{-1})\ |+;1\rangle\ , \tag{5.20}$$

$$\Phi_+^{(0,1)}(\zeta)\ |+;1\rangle = \Lambda(\zeta;v^2)\frac{\varphi(\zeta^{-2};r)}{\varphi(\zeta^2;r)}\ \left(\frac{(\zeta^2-v^2)}{(1-v^2\zeta^2)}\Phi_+^{(0,1)}(\zeta^{-1})\ - k'_+\frac{v^2\zeta(\zeta^2-\zeta^{-2})}{1-v^2\zeta^2}\ \Phi_-^{(0,1)}(\zeta^{-1})\ \right)\ |+;1\rangle\ . \tag{5.21}$$

By analogy with the case of diagonal boundary conditions, it is straigthforward to derive the action of the conserved currents (3.11) for $k_-=0$ on above states. Appart from terms that already appeared in the case of diagonal boundary conditions, the structure of the states $|+;i\rangle$ generates an additional contribution associated with the currents $\overline{\mathcal{W}}_\pm^{(i)}(\zeta)$ which mixes with the one associated with $\overline{\mathcal{Z}}_+^{(i)}(\zeta)$, $i=0,1$. Assuming that $|+;i\rangle$ are eigenstates of (3.11) for $k_-=0$ determines uniquely the choice of parameters k'_+, k''_+. Namely,

$$\begin{aligned}\overline{\mathcal{I}}^{(0)}(\zeta)|_{k_-=0}|+;0\rangle &= \Lambda^{(0)}(\zeta^2,r)\frac{\varphi(\zeta^{-2},r)}{\varphi(\zeta^2,r)}\frac{(\epsilon_+\zeta^{-1}+\epsilon_-\zeta)}{g(\zeta^2-\zeta^{-2})}|+;0\rangle\ ,\\ \overline{\mathcal{I}}^{(1)}(\zeta)|_{k_-=0}|+;1\rangle &= \Lambda^{(1)}(\zeta^2,r)\frac{\varphi(\zeta^{-2},r)}{\varphi(\zeta^2,r)}\frac{(\epsilon_+\zeta^{-1}+\epsilon_-\zeta)}{g(\zeta^2-\zeta^{-2})}|+;1\rangle\end{aligned}$$

for

$$k''_+ = \frac{1}{q-q^{-1}}\frac{k_+}{\epsilon_-} \qquad \text{and} \qquad k'_+ = -\frac{1}{q-q^{-1}}\frac{k_+}{\epsilon_+}\ . \tag{5.22}$$

Although the vacuum vectors (5.17) with (5.22) are more complicated than in the diagonal case, the spectrum of the transfer matrix is clearly unchanged, as already pointed out in [BCRS] (see also [MMR]) within the Bethe ansatz framework. Having identified the vacuum vectors, excited states follow using the action of type II $q-$vertex operators.

For completeness, let us finally describe the vacuum eigenstates of the Hamiltonian (3.1) for lower non-diagonal boundary conditions $\epsilon_\pm\neq 0, k_-\neq 0$ and $k_+=0$. They are given by:

$$|-;0\rangle = \sum_{n=0}^{\infty}\frac{q^{-n(n-1)/2}}{[n]_q!}(\overline{\mathsf{w}}_1^{(-)})^n|0\rangle_B \qquad \text{and} \qquad |-;1\rangle = \sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{[n]_q!}(\overline{\mathsf{w}}_0^{(-)})^n|1\rangle_B \tag{5.23}$$

where

$$k''_- = -\frac{1}{q-q^{-1}}\frac{k_-}{\epsilon_-} \qquad \text{and} \qquad k'_- = \frac{1}{q-q^{-1}}\frac{k_-}{\epsilon_+}\ . \tag{5.24}$$

Note that the vacuum vectors (5.17) with (5.22) and (5.23) with (5.24) are power series in k_+ and k_-, respectively. For the special case $k_+=0$ in (5.17) (or $k_-=0$ in (5.23)), all terms in the series disappear except the term $n=0$, in which case the vacuum vectors reduce to the ones associated with diagonal boundary conditions $k_\pm=0$.

6. Comments and perspectives

In the present article, the research program initiated in [Bas1, Bas2] and further explored in [BK1, BK2, BK3] has been applied to the thermodynamic limit of the XXZ open spin chain with general integrable boundary conditions. It has been shown that the formulation of the finite size case of [BK3] - here completed for diagonal boundary conditions - can be directly extended to the infinite limit. In this approach, the new current algebra introduced in [BSh] plays a central role: the diagonalisation of the transfer matrix is reduced to the study of the representation theory of the current algebra $O_q(\widehat{sl_2})$. For $-1<q<0$, explicit realizations of the currents in terms of $U_q(\widehat{sl_2})$ $q-$vertex operators

have been obtained, confirming independently the proposal of [BB2]. Also, certain properties reminiscent of the finite size case - for instance, the existence of two 'dual' families of eigenstates - have been exhibited. For diagonal boundary conditions, the spectrum and eigenstates have been described in details within the new framework, providing a fresh look at the known results in [JKKKM]. For upper or lower non-diagonal boundary, the eigenstates are also generated starting from the currents' eigenstates through the action of Chevalley elements of $U_q(\widehat{sl_2})$.

For generic boundary conditions, the spectral problem could be considered along the same line: equations extending (5.18)-(5.21) are then considered, where, roughly speaking, the eigenstates are such that an additional term is generated in the r.h.s. of (5.18) and (5.20). Actually, in view of the fact that the elements (5.16) generate a $q-$Onsager algebra, according to the intertwining relations of the form (4.2) it is thus natural to construct the eigenstates of (3.1) using linearly independent monomials in $\overline{\mathsf{w}}_0, \overline{\mathsf{w}}_1$. Namely, understanding further the construction of a Poincaré-Birkhoff-Witt basis of the $q-$Onsager algebra is highly desirable. In this direction, for certain reasons that will not be detailed here, the results of [BB3] then suggest to consider the following combinations of 'descendents' acting on the 'diagonal' vacuum vectors:

$$\mathcal{W}_{-k_1}^{\alpha_1}...\mathcal{W}_{-k_N}^{\alpha_N}\mathcal{G}_{p_1+1}^{\beta_1}...\mathcal{G}_{p_P+1}^{\beta_P}\mathcal{W}_{l_M+1}^{\gamma_M}...\mathcal{W}_{l_1+1}^{\gamma_1}|i\rangle_B \tag{6.1}$$

where $\{\alpha_j,\beta_j,k_j,l_j,r_j\}\in\mathbb{Z}_+$ and the ordering; $k_1<...<k_N$; $l_1<...<l_M$; $p_1<...<p_P$ is chosen. We intend to study this problem separately.

As the reader noticed, the long standing question of the non-Abelian symmetry of the Hamiltonian (3.1) has not been rised up to now although it played a central role in the initial development of the vertex operator program [DFJMN, JKKKM]: in the case of the infinite XXZ spin chain, recall that the $U_q(\widehat{sl_2})$ algebra emerges as a non-Abelian symmetry of the Hamiltonian [FM]. For generic *non-diagonal* or *diagonal* boundary conditions, following [FM, Ji] it is easy to show that a similar phenomena occurs in the thermodynamic limit of the open spin chain. Let $H_{\frac{1}{2}XXZ}=H_0+h_b$ where H_0 and h_b denote the bulk and boundary contributions in the Hamiltonian (3.1) for $\epsilon_\pm\neq 0$, $k_\pm\neq 0$, respectively. By straightforward calculations one finds:

$$\big[H_0,\mathcal{W}_0^{(\infty)}\big] \;=\; -\big[h_b,\mathcal{W}_0^{(\infty)}\big]=-\frac{1}{2}(q-q^{-1})\big(...\otimes q^{\sigma_3}\otimes q^{\sigma_3}\otimes(k_+\sigma_+-k_-\sigma_-)\big)\ .$$

A similar analysis can be done for $\mathcal{W}_1^{(\infty)}$. Combining both expressions, one finally shows that the Hamiltonian (3.1) is commuting with these operators:

$$\big[H_{\frac{1}{2}XXZ},a\big]=0\ ,\qquad a\in\{\mathcal{W}_0^{(\infty)},\mathcal{W}_1^{(\infty)}\}\ . \tag{6.2}$$

Similarly, for the case of generic *diagonal* boundary conditions, in the thermodynamic limit $N\rightarrow\infty$ by straightforward calculations one finds that the contributions coming from the commutator of the bulk and boundary terms of the Hamiltonian with the fundamental elements of $\mathcal{A}_q^{diag}$ cancel each other:

$$\big[H_{\frac{1}{2}XXZ}^{diag},a\big]=0\ ,\qquad a\in\{\mathcal{K}_0^{(\infty)},\mathcal{K}_1^{(\infty)},\mathcal{Z}_1^{(\infty)},\tilde{\mathcal{Z}}_1^{(\infty)}\}\ . \tag{6.3}$$

According to these results, we then conclude that the $q-$Onsager and augmented $q-$Onsager algebras with defining relations (2.16) and (2.18) emerge as the non-Abelian symmetry of the Hamiltonian (3.1) for generic *non-diagonal* and *diagonal* boundary conditions ($k_\pm=0$), respectively. Despite of the fact that this property played no role in previous analysis, it has not been observed previously in the literature, to our knowledge.

Besides, we would like to make a few comments. In [PS], recall that common algebraic structures were exhibited between certain finite lattice models and conformal field theories. For instance, it was shown that the Hamiltonian of the XXZ open spin chain can be understood as the discrete analog of the Virasoro generator L_0. In this picture, the Temperley-Lieb algebra and the Virasoro algebra share similar properties. For instance, consider the Hamiltonians

$$H_{\frac{1}{2}XXZ}^{(\pm)} \;=\; -\frac{1}{2}\sum_{k=1}^{\infty}\Big(\sigma_1^{k+1}\sigma_1^k+\sigma_2^{k+1}\sigma_2^k+\Delta\sigma_3^{k+1}\sigma_3^k\Big)\pm\frac{(q-q^{-1})}{4}\sigma_3^1 \tag{6.4}$$

which can be obtained as the thermodynamic limit of the $U_q(sl_2)-$symmetric XXZ open spin chain. As described in [PS], the special value of the boundary field (compared with (3.1)) plays a very singular role: for the deformation parameter $q=\exp(i\pi/\mu(\mu+1))$, $\mu\notin\mathbb{Q}$, the central charge of the Virasoro algebra associated with $H_{\frac{1}{2}XXZ}^{(-)}$ was

identified with $c = 1 - 6/\mu(\mu+1)$ and the Hamiltonian's spectrum was expressed in terms of conformal dimensions. In light of previous results, for the special class of diagonal boundary conditions $(+)$ (resp. $(-)$) associated with $\epsilon_+ = 0, \epsilon_- \neq 0$ (resp. $\epsilon_- = 0, \epsilon_+ \neq 0$) some remarkable properties are then expected. Indeed, according to the analysis above, the vacuum vector of the Hamiltonian $H^{(+)}_{\frac{1}{2}XXZ}$ (resp. $H^{(-)}_{\frac{1}{2}XXZ}$) is given by $|B_+\rangle$ (resp. $|B_-\rangle$). In other words, the spectrum of the Hamiltonians (6.4) is classified according to the eigenvalues of the fundamental generators of the augmented $q-$Onsager algebra. Moreover, it is worth mentioning that the realizations of the $O_q(\widehat{sl_2})$ currents in terms of type II $q-$vertex operators such as (4.12) share some analogy with currents arising in the study of the $q-$deformed Virasoro algebra [LP] or currents exihibited in the context of conformal field theory [Kau]. In view of this, a relation between the representation theory of $O_q(\widehat{sl_2})$ and the $q-$Virasoro algebra may be investigated.

Finally, let us mention that the formulation (3.12) can be applied to other integrable models directly, for instance the XXZ open chain with higher spins or alternating spins. In these cases, the results of [IIJMNT] have to be considered. More generally, it can be extended to models with higher symmetries (see e.g. [BR]) in which case generalizations of the current algebra (3.3)-(3.8) are needed. A first step in this direction has been passed in [BB1, Kol], where generalizations of the $q-$Onsager algebra and twisted $q-$ Yangians [MRS] have been proposed (for some applications, see [BF]). In this picture, the problem of the diagonalization of the transfer matrix generalizing (3.12) with (3.11) relies on a better understanding of the representation theory associated with certain coideal subalgebras of $U_q(\widehat{g})$. Another interesting direction, obviously inspired by the conformal field theory program, concerns the family of $q-$difference equations for the correlation functions in Onsager's picture. We intend to discuss some of these problems elsewhere.

Acknowledgements: P.B. thanks T. Kojima for discussions. S.B. thanks N. Crampé for discussions and also the LMPT for hospitality, where part of this work has been done.

APPENDIX A: Generators of the infinite dimensional algebras $\mathcal{A}_q$ and $\mathcal{A}_q^{diag}$

• **Elements generating $\mathcal{A}_q$:** For generic values of the parameters $\epsilon_\pm$, $k_\pm \neq 0$ and $N \in \mathbb{N}$, the elements $\mathsf{W}_{-k}, \mathsf{W}_{k+1}, \mathsf{G}_{k+1}, \tilde{\mathsf{G}}_{k+1}$ ($4N$ in total) act on $N-$tensor product (evaluation) representations of $U_q(\widehat{sl_2})$, and depend solely on the $N-$parameters v_k and spin$-j_k$ for $k = 1, ..., N$. Define $w_0^{(j_k)} = q^{2j_k+1} + q^{-2j_k-1}$. According to the ordering of the vector spaces

$$\mathcal{V}^{(N)} = \mathcal{V}_N \otimes \cdots \otimes \mathcal{V}_2 \otimes \mathcal{V}_1 \ , \tag{6.5}$$

they act as [BK2] (see also [BK1]):

$$\begin{aligned}
\mathsf{W}_{-k}^{(N)} &= \frac{(w_0^{(j_N)} - (q+q^{-1})q^{2s_3})}{(q+q)} \otimes \mathsf{W}_k^{(N-1)} - \frac{(v_N^2 + v_N^{-2})}{(q+q^{-1})} I\!I \otimes \mathsf{W}_{-k+1}^{(N-1)} + \frac{(v_N^2+v_N^{-2})w_0^{(j_N)}}{(q+q^{-1})^2}\mathsf{W}_{-k+1}^{(N)} \\
&\quad + \frac{(q-q^{-1})}{k_+k_-(q+q)^2}\Big(k_+v_Nq^{1/2}S_+q^{s_3} \otimes \mathsf{G}_k^{(N-1)} + k_-v_N^{-1}q^{-1/2}S_-q^{s_3} \otimes \tilde{\mathsf{G}}_k^{(N-1)}\Big) \\
&\quad + q^{2s_3} \otimes \mathsf{W}_{-k}^{(N-1)} \ , \\
\mathsf{W}_{k+1}^{(N)} &= \frac{(w_0^{(j_N)} - (q+q^{-1})q^{-2s_3})}{(q+q^{-1})} \otimes \mathsf{W}_{-k+1}^{(N-1)} - \frac{(v_N^2 + v_N^{-2})}{(q+q^{-1})} I\!I \otimes \mathsf{W}_k^{(N-1)} + \frac{(v_N^2+v_N^{-2})w_0^{(j_N)}}{(q+q^{-1})^2}\mathsf{W}_k^{(N)} \\
&\quad + \frac{(q-q^{-1})}{k_+k_-(q+q^{-1})^2}\Big(k_+v_N^{-1}q^{-1/2}S_+q^{-s_3} \otimes \mathsf{G}_k^{(N-1)} + k_-v_Nq^{1/2}S_-q^{-s_3} \otimes \tilde{\mathsf{G}}_k^{(N-1)}\Big) \\
&\quad + q^{-2s_3} \otimes \mathsf{W}_{k+1}^{(N-1)} \ , \\
\mathsf{G}_{k+1}^{(N)} &= \frac{k_-(q-q^{-1})^2}{k_+(q+q^{-1})}S_-^2 \otimes \tilde{\mathsf{G}}_k^{(N-1)} - \frac{1}{(q+q^{-1})}(v_N^2q^{2s_3} + v_N^{-2}q^{-2s_3}) \otimes \mathsf{G}_k^{(N-1)} + I\!I \otimes \mathsf{G}_{k+1}^{(N-1)} \\
&\quad +(q-q^{-1})\Big(k_-v_Nq^{-1/2}S_-q^{s_3} \otimes \big(\mathsf{W}_{-k}^{(N-1)} - \mathsf{W}_k^{(N-1)}\big) + k_-v_N^{-1}q^{1/2}S_-q^{-s_3} \otimes \big(\mathsf{W}_{k+1}^{(N-1)} - \mathsf{W}_{-k+1}^{(N-1)}\big)\Big) \\
&\quad + \frac{(v_N^2+v_N^{-2})w_0^{(j_N)}}{(q+q^{-1})^2}\mathsf{G}_k^{(N)} \ , \\
\tilde{\mathsf{G}}_{k+1}^{(N)} &= \frac{k_+(q-q^{-1})^2}{k_-(q+q^{-1})}S_+^2 \otimes \mathsf{G}_k^{(N-1)} - \frac{1}{(q+q^{-1})}(v_N^2q^{-2s_3} + v_N^{-2}q^{2s_3}) \otimes \tilde{\mathsf{G}}_k^{(N-1)} + I\!I \otimes \tilde{\mathsf{G}}_{k+1}^{(N-1)} \\
&\quad +(q-q^{-1})\Big(k_+v_N^{-1}q^{1/2}S_+q^{s_3} \otimes \big(\mathsf{W}_{-k}^{(N-1)} - \mathsf{W}_k^{(N-1)}\big) + k_+v_Nq^{-1/2}S_+q^{-s_3} \otimes \big(\mathsf{W}_{k+1}^{(N-1)} - \mathsf{W}_{-k+1}^{(N-1)}\big)\Big) \\
&\quad + \frac{(v_N^2+v_N^{-2})w_0^{(j_N)}}{(q+q^{-1})^2}\tilde{\mathsf{G}}_k^{(N)} \ ,
\end{aligned} \tag{6.6}$$

where, for the special case $k = 0$ we identify [16]

$$\mathsf{W}_k^{(N)}|_{k=0} \equiv 0 \ , \quad \mathsf{W}_{-k+1}^{(N)}|_{k=0} \equiv 0 \ , \quad \mathsf{G}_k^{(N)}|_{k=0} = \tilde{\mathsf{G}}_k^{(N)}|_{k=0} \equiv \frac{k_+k_-(q+q^{-1})^2}{(q-q^{-1})} I\!I^{(N)} \ . \tag{6.7}$$

In addition, one has the "initial" $c-$number conditions

$$\mathsf{W}_0^{(0)} \equiv \epsilon_+^{(0)} \ , \quad \mathsf{W}_1^{(0)} \equiv \epsilon_-^{(0)} \qquad \text{and} \qquad \mathsf{G}_1^{(0)} = \tilde{\mathsf{G}}_1^{(0)} \equiv \epsilon_+^{(0)}\epsilon_-^{(0)}(q-q^{-1}) \ . \tag{6.8}$$

• **Elements generating $\mathcal{A}_q^{diag}$:** All expressions below are derived from above expressions, through the substitutions:

$$\begin{aligned}
&\mathsf{W}_{-k}^{(N)} \rightarrow \mathsf{K}_{-k}^{(N)} \ , \quad \mathsf{W}_{k+1}^{(N)} \rightarrow \mathsf{K}_{k+1}^{(N)} \ , \\
&\mathsf{G}_{k+1}^{(N)} \rightarrow k_-\big(\mathsf{Z}_{k+1}^{(N)} + \epsilon_+\epsilon_-(q-q^{-1}) \ I\!I^{(N)}\big) \ , \quad \tilde{\mathsf{G}}_{k+1}^{(N)} \rightarrow k_+\big(\tilde{\mathsf{Z}}_{k+1}^{(N)} + \epsilon_+\epsilon_-(q-q^{-1}) \ I\!I^{(N)}\big)
\end{aligned}$$

[16]Although the notation is ambiguous, the reader must keep in mind that $\mathsf{W}_k^{(N)}|_{k=0} \neq \mathsf{W}_{-k}^{(N)}|_{k=0}$,$\mathsf{W}_{-k+1}^{(N)}|_{k=0} \neq \mathsf{W}_{k+1}^{(N)}|_{k=0}$ for any N.

and then setting $k_\pm = 0$. It yields to:

$$
\begin{aligned}
\mathsf{K}^{(N)}_{-k} &= \frac{(w_0^{(j_N)} - (q+q^{-1})q^{s_3})}{(q+q^{-1})} \otimes \mathsf{K}^{(N-1)}_{k} - \frac{(v_N^2 + v_N^{-2})}{(q+q^{-1})} I\!I \otimes \mathsf{K}^{(N-1)}_{-k+1} + \frac{(v_N^2+v_N^{-2})w_0^{(j_N)}}{(q+q^{-1})^2}\mathsf{K}^{(N)}_{-k+1} \\
&\quad + \frac{(q-q^{-1})}{(q+q^{-1})^2}\Big(v_N q^{1/2} S_+ q^{s_3} \otimes \mathsf{Z}^{(N-1)}_{k} + v_N^{-1} q^{-1/2} S_- q^{s_3} \otimes \tilde{\mathsf{Z}}^{(N-1)}_{k}\Big) + q^{2s_3} \otimes \mathsf{K}^{(N-1)}_{-k} , \\
\mathsf{K}^{(N)}_{k+1} &= \frac{(w_0^{(j_N)} - (q+q^{-1})q^{-s_3})}{(q+q^{-1})} \otimes \mathsf{K}^{(N-1)}_{-k+1} - \frac{(v_N^2 + v_N^{-2})}{(q+q^{-1})} I\!I \otimes \mathsf{K}^{(N-1)}_{k} + \frac{(v_N^2+v_N^{-2})w_0^{(j_N)}}{(q+q^{-1})^2}\mathsf{K}^{(N)}_{k} \\
&\quad + \frac{(q-q^{-1})}{(q+q^{-1})^2}\Big(v_N^{-1} q^{-1/2} S_+ q^{-s_3} \otimes \mathsf{Z}^{(N-1)}_{k} + v_N q^{1/2} S_- q^{-s_3} \otimes \tilde{\mathsf{Z}}^{(N-1)}_{k}\Big) + q^{-2s_3} \otimes \mathsf{K}^{(N-1)}_{k+1} , \\
\mathsf{Z}^{(N)}_{k+1} &= \frac{(q-q^{-1})^2}{(q+q^{-1})} S_-^2 \otimes \tilde{\mathsf{Z}}^{(N-1)}_{k} - \frac{1}{(q+q^{-1})}(v_N^2 q^{2s_3} + v_N^{-2} q^{-2s_3}) \otimes \mathsf{Z}^{(N-1)}_{k} + I\!I \otimes \mathsf{Z}^{(N-1)}_{k+1} \\
&\quad + (q-q^{-1})\Big(v_N q^{-1/2} S_- q^{s_3} \otimes \big(\mathsf{K}^{(N-1)}_{-k} - \mathsf{K}^{(N-1)}_{k}\big) + v_N^{-1} q^{1/2} S_- q^{-s_3} \otimes \big(\mathsf{K}^{(N-1)}_{k+1} - \mathsf{K}^{(N-1)}_{-k+1}\big)\Big) \\
&\quad + \frac{(v_N^2+v_N^{-2})w_0^{(j_N)}}{(q+q^{-1})^2}\mathsf{Z}^{(N)}_{k} , \\
\tilde{\mathsf{Z}}^{(N)}_{k+1} &= \frac{(q-q^{-1})^2}{(q+q^{-1})} S_+^2 \otimes \mathsf{Z}^{(N-1)}_{k} - \frac{1}{(q+q^{-1})}(v_N^2 q^{-2s_3} + v_N^{-2} q^{2s_3}) \otimes \tilde{\mathsf{Z}}^{(N-1)}_{k} + I\!I \otimes \tilde{\mathsf{Z}}^{(N-1)}_{k+1} \\
&\quad + (q-q^{-1})\Big(v_N^{-1} q^{1/2} S_+ q^{s_3} \otimes \big(\mathsf{K}^{(N-1)}_{-k} - \mathsf{K}^{(N-1)}_{k}\big) + v_N q^{-1/2} S_+ q^{-s_3} \otimes \big(\mathsf{K}^{(N-1)}_{k+1} - \mathsf{K}^{(N-1)}_{-k+1}\big)\Big) \\
&\quad + \frac{(v_N^2+v_N^{-2})w_0^{(j_N)}}{(q+q^{-1})^2}\tilde{\mathsf{Z}}^{(N)}_{k} .
\end{aligned} \tag{6.9}
$$

As before, we identify [17]

$$
\mathsf{K}^{(N)}_{k}|_{k=0} \equiv 0 , \quad \mathsf{K}^{(N)}_{-k+1}|_{k=0} \equiv 0 , \quad \mathsf{Z}^{(N)}_{k}|_{k=0} \equiv 0 , \quad \tilde{\mathsf{Z}}^{(N)}_{k}|_{k=0} \equiv 0 \tag{6.10}
$$

together with the "initial" $c-$number conditions

$$
\mathsf{K}^{(0)}_{0} \equiv \epsilon_+ , \quad \mathsf{K}^{(0)}_{1} \equiv \epsilon_- \qquad \text{and} \qquad \mathsf{Z}^{(0)}_{1} = \tilde{\mathsf{Z}}^{(0)}_{1} \equiv 0 . \tag{6.11}
$$

• **Application to the homogeneous XXZ open spin-$\frac{1}{2}$ chain:** For generic non-diagonal $k_\pm \neq 0$ or diagonal $k_\pm \equiv 0$ boundary conditions, the generators of the infinite dimensional algebras $\mathcal{A}_q$ or $\mathcal{A}_q^{diag}$ are simply given, respectively, by:

$$
\begin{aligned}
\mathcal{W}^{(N)}_{-l} &\equiv (\otimes_{k=1}^{N}\pi^{(\frac{1}{2})})[\mathsf{W}^{(N)}_{-l}]|_{v_k=1} , \qquad \mathcal{W}^{(N)}_{l+1} \equiv (\otimes_{k=1}^{N}\pi^{(\frac{1}{2})})[\mathsf{W}^{(N)}_{l+1}]|_{v_k=1} , \\
\mathcal{G}^{(N)}_{l+1} &\equiv (\otimes_{k=1}^{N}\pi^{(\frac{1}{2})})[\mathsf{G}^{(N)}_{l+1}]|_{v_k=1} , \qquad \tilde{\mathcal{G}}^{(N)}_{l+1} \equiv (\otimes_{k=1}^{N}\pi^{(\frac{1}{2})})[\tilde{\mathsf{G}}^{(N)}_{l+1}]|_{v_k=1}
\end{aligned}
$$

or

$$
\begin{aligned}
\mathcal{K}^{(N)}_{-l} &\equiv (\otimes_{k=1}^{N}\pi^{(\frac{1}{2})})[\mathsf{K}^{(N)}_{-l}]|_{v_k=1} , \qquad \mathcal{K}^{(N)}_{l+1} \equiv (\otimes_{k=1}^{N}\pi^{(\frac{1}{2})})[\mathsf{K}^{(N)}_{l+1}]|_{v_k=1} , \\
\mathcal{Z}^{(N)}_{l+1} &\equiv (\otimes_{k=1}^{N}\pi^{(\frac{1}{2})})[\mathsf{Z}^{(N)}_{l+1}]|_{v_k=1} , \qquad \tilde{\mathcal{Z}}^{(N)}_{l+1} \equiv (\otimes_{k=1}^{N}\pi^{(\frac{1}{2})})[\tilde{\mathsf{Z}}^{(N)}_{l+1}]|_{v_k=1}
\end{aligned} \tag{6.12}
$$

for $l \in 0, ..., N-1$. Here we considered the two-dimensional representation $\pi^{(1/2)}$ given by:

$$
\pi^{(1/2)}[S_\pm] = \sigma_\pm \qquad \text{and} \qquad \pi^{(1/2)}[s_3] = \sigma_3/2 . \tag{6.13}
$$

[17] Remind that $\mathsf{K}^{(N)}_{k}|_{k=0} \neq \mathsf{K}^{(N)}_{-k}|_{k=0}$,$\mathsf{K}^{(N)}_{-k+1}|_{k=0} \neq \mathsf{K}^{(N)}_{k+1}|_{k=0}$ for any N.

APPENDIX B: Drinfeld-Jimbo presentation of $U_q(\widehat{sl_2})$

Define the extended Cartan matrix $\{a_{ij}\}$ ($a_{ii}=2, a_{ij}=-2$ for $i\neq j$). The quantum affine algebra $U_q(\widehat{sl_2})$ is generated by the elements $\{h_j, e_j, f_j\}$, $j\in\{0,1\}$ which satisfy the defining relations

$$[h_i,h_j]=0\ ,\quad [h_i,e_j]=a_{ij}e_j\ ,\quad [h_i,f_j]=-a_{ij}f_j\ ,\quad [e_i,f_j]=\delta_{ij}\frac{q^{h_i}-q^{-h_i}}{q-q^{-1}}$$

together with the $q-$Serre relations

$$[e_i,[e_i,[e_i,e_j]_q]_{q^{-1}}]=0\ ,\quad \text{and}\quad [f_i,[f_i,[f_i,f_j]_q]_{q^{-1}}]=0\ . \tag{6.14}$$

The sum $C=h_0+h_1$ is the central element of the algebra. The Hopf algebra structure is ensured by the existence of a comultiplication $\Delta: U_q(\widehat{sl_2})\mapsto U_q(\widehat{sl_2})\otimes U_q(\widehat{sl_2})$, antipode $\mathcal{S}: U_q(\widehat{sl_2})\mapsto U_q(\widehat{sl_2})$ and a counit $\mathcal{E}: U_q(\widehat{sl_2})\mapsto \mathbb{C}$ with

$$\begin{aligned}\Delta(e_i) &= e_i\otimes 1+q^{h_i}\otimes e_i\ ,\\ \Delta(f_i) &= f_i\otimes q^{-h_i}+1\otimes f_i\ ,\\ \Delta(h_i) &= h_i\otimes 1+1\otimes h_i\ ,\end{aligned} \tag{6.15}$$

$$\mathcal{S}(e_i)=-q^{-h_i}e_i\ ,\quad \mathcal{S}(f_i)=-f_iq^{h_i}\ ,\quad \mathcal{S}(h_i)=-h_i\qquad \mathcal{S}(1)=1$$

and

$$\mathcal{E}(e_i)=\mathcal{E}(f_i)=\mathcal{E}(h_i)=0\ ,\qquad \mathcal{E}(1)=1\ .$$

Note that the opposite coproduct Δ' can be similarly defined with $\Delta'\equiv\sigma\circ\Delta$ where the permutation map $\sigma(x\otimes y)=y\otimes x$ for all $x,y\in U_q(\widehat{sl_2})$ is used.

The (evaluation in the principal gradation) endomorphism $\pi_\zeta: U_q(\widehat{sl_2})\mapsto \mathrm{End}(\mathcal{V}_\zeta)$ is chosen such that ($\mathcal{V}\equiv\mathbb{C}^2$)

$$\begin{aligned}\pi_\zeta[e_1]&=\zeta\sigma_+\ , & \pi_\zeta[e_0]&=\zeta\sigma_-\ ,\\ \pi_\zeta[f_1]&=\zeta^{-1}\sigma_-\ , & \pi_\zeta[f_0]&=\zeta^{-1}\sigma_+\ ,\\ \pi_\zeta[q^{h_1}]&=q^{\sigma_3}\ , & \pi_\zeta[q^{h_0}]&=q^{-\sigma_3}\ ,\end{aligned} \tag{6.16}$$

in terms of the Pauli matrices $\sigma_\pm,\sigma_3$:

$$\sigma_+=\left(\begin{array}{cc}0&1\\0&0\end{array}\right)\ ,\qquad \sigma_-=\left(\begin{array}{cc}0&0\\1&0\end{array}\right)\ ,\qquad \sigma_3=\left(\begin{array}{cc}1&0\\0&-1\end{array}\right)\ .$$

The $R-$matrix here considered is the solution of the intertwining equation:

$$R(\zeta_1/\zeta_2)(\pi_{\zeta_1}\otimes\pi_{\zeta_2})\,\Delta(x)=(\pi_{\zeta_1}\otimes\pi_{\zeta_2})\,(\sigma\circ\Delta(x))R(\zeta_1/\zeta_2)\ .$$

According to above definitions and up to an overall scalar factor, in the principal picture it reads:

$$R(\zeta)=\frac{1}{\kappa(\zeta)}\left(\begin{array}{cccc}1&0&0&0\\ 0&\frac{(1-\zeta^2)q}{1-q^2\zeta^2}&\frac{(1-q^2)\zeta}{1-q^2\zeta^2}&0\\ 0&\frac{(1-q^2)\zeta}{1-q^2\zeta^2}&\frac{(1-\zeta^2)q}{1-q^2\zeta^2}&0\\ 0&0&0&1\end{array}\right)\ . \tag{6.17}$$

where the scalar factor

$$\kappa(\zeta)=\zeta\frac{(q^4\zeta^2;q^4)_\infty(q^2\zeta^{-2};q^4)_\infty}{(q^4\zeta^{-2};q^4)_\infty(q^2\zeta^2;q^4)_\infty}\ ,\qquad (z;p)_\infty=\prod_{n=0}^{\infty}(1-zp^n) \tag{6.18}$$

is chosen to ensure unitarity and crossing symmetry of the $R-$matrix:

$$\begin{aligned}R(\zeta)R(\zeta^{-1}) &= I\!I\otimes I\!I\ ,\\ R^{\epsilon_2'\epsilon_1}_{\epsilon_2\epsilon_1'}(\zeta^{-1}) &= R^{-\epsilon_1'\epsilon_2'}_{-\epsilon_1\epsilon_2}(-q^{-1}\zeta)\ .\end{aligned} \tag{6.19}$$

APPENDIX C: The $q-$Vertex operators of $U_q(\widehat{sl_2})$

The so-called type I and type II $q-$vertex operators satisfy the commutation relations:

(6.20) $$\Phi_{\epsilon_2}(\zeta_2)\Phi_{\epsilon_1}(\zeta_1)=\sum_{\epsilon'_1,\epsilon'_2}R_{\epsilon_1\,\epsilon_2}^{\epsilon'_1\,\epsilon'_2}(\zeta_1/\zeta_2)\Phi_{\epsilon'_1}(\zeta_1)\Phi_{\epsilon'_2}(\zeta_2)\ ,$$

(6.21) $$\Psi^*_{\mu'_1}(\zeta_1)\Psi^*_{\mu'_2}(\zeta_2)=-\sum_{\mu_1,\mu_2}R_{\mu_1\,\mu_2}^{\mu'_1\,\mu'_2}(\zeta_1/\zeta_2)\Psi^*_{\mu_2}(\zeta_2)\Psi^*_{\mu_1}(\zeta_1)\ ,$$

(6.22) $$\Phi_\epsilon(\zeta_1)\Psi^*_\mu(\zeta_2)=\tau(\zeta_1/\zeta_2)\Psi^*_\mu(\zeta_2)\Phi_\epsilon(\zeta_1)\ .$$

Here
$$\tau(\zeta)=\zeta^{-1}\frac{\Theta_{q^4}(q\zeta^2)}{\Theta_{q^4}(q\zeta^{-2})}\ ,\qquad \Theta_p(z)=(z;p)_\infty(pz^{-1};p)_\infty(p;p)_\infty\ .$$

Define $\Phi^*_\epsilon(\zeta)=\Phi_{-\epsilon}(-q^{-1}\zeta)$. The type I vertex operators satisfy the invertibility relations

(6.23) $$g\sum_\epsilon\Phi^*_\epsilon(\zeta)\Phi_\epsilon(\zeta)=\mathrm{id}\ ,\qquad g\,\Phi_{\epsilon_1}(\zeta)\Phi^*_{\epsilon_2}(\zeta)=\delta_{\epsilon_1\epsilon_2}\mathrm{id}\quad\text{with}\quad g=\frac{\big(q^2;q^4\big)_\infty}{\big(q^4;q^4\big)_\infty}\ .$$

Type I and type II $q-$vertex operators admit a bosonic realization [DFJMN]. For $i=0,1$, consider the bosonic Fock space
$$\mathcal{H}^{(i)}=\mathbf{C}[a_{-1},a_{-2},\cdots]\otimes\left(\oplus_{n\in\mathbf{Z}}\mathbf{C}e^{\Lambda_i+n\alpha}\right)$$
where the commutation relations of a_n are given by
$$[a_m,a_n]=\delta_{m+n,0}\frac{[m][2m]}{m}\ ,\quad\text{with}\qquad m,n\neq0\quad\text{and}\qquad [n]=\frac{q^n-q^{-n}}{q-q^{-1}}\ .$$
Define $[\partial,\alpha]=2$, $[\partial,\Lambda_0]=0$ and $\Lambda_1=\Lambda_0+\alpha/2$. The highest weight vector of $\mathcal{H}^{(i)}$ is given by $|i\rangle=1\otimes e^{\Lambda_i}$ and the operators e^β, z^∂ act as
$$e^\beta.e^\gamma=e^{\beta+\gamma},\qquad z^\partial.e^\gamma=z^{[\partial,\gamma]}e^\gamma\ .$$

The bosonic realization for the type I and type II $q-$vertex operators reads [DFJMN]:

(6.24) $$\Phi_-^{(1-i,i)}(\zeta)\ =\ e^{P(\zeta^2)}e^{Q(\zeta^2)}\otimes e^{\alpha/2}(-q^3\zeta^2)^{(\partial+i)/2}\zeta^{-i}\ ,$$
$$\Phi_+^{(1-i,i)}(\zeta)\ =\ \oint_{C_1}\frac{dw}{2\pi i}\frac{(1-q^2)w\zeta}{q(w-q^2\zeta^2)(w-q^4\zeta^2)}:\Phi_-^{(1-i,i)}(\zeta)X^-(w):\ ,$$

(6.25)

(6.26) $$\Psi_-^{*(1-i,i)}(\zeta)\ =\ e^{-P(q^{-1}\zeta^2)}e^{-Q(q\zeta^2)}\otimes e^{-\alpha/2}(-q^3\zeta^2)^{(-\partial+i)/2}\zeta^{1-i}\ ,$$

(6.27) $$\Psi_+^{*(1-i,i)}(\zeta)\ =\ \oint_{C_2}\frac{dw}{2\pi i}\frac{q^2(1-q^2)\zeta}{(w-q^2\zeta^2)(w-q^4\zeta^2)}:\Psi_-^{*(1-i,i)}(\zeta)X^+(w):\ ,$$

where

(6.28) $$X^\pm(z)\ =\ e^{R^\pm(z)}e^{S^\pm(z)}\otimes e^{\pm\alpha}z^{\pm\partial}\ ,$$

$$P(z)=\sum_{n=1}^\infty\frac{a_{-n}}{[2n]}q^{7n/2}z^n\ ,\qquad Q(z)=-\sum_{n=1}^\infty\frac{a_n}{[2n]}q^{-5n/2}z^{-n}\ ,$$
$$R^\pm(z)=\pm\sum_{n=1}^\infty\frac{a_{-n}}{[n]}q^{\mp n/2}z^n\ ,\quad S^\pm(z)=\mp\sum_{n=1}^\infty\frac{a_n}{[n]}q^{\mp n/2}z^{-n}\ .$$

The integration contours encircle $w=0$ in such a way that
$$C_1\quad:\quad q^4\zeta^2\ \text{is inside and}\ q^2\zeta^2\ \text{is outside,}$$
$$C_2\quad:\quad q^4\zeta^2\ \text{is outside and}\ q^2\zeta^2\ \text{is inside.}$$

References

[ABBBQ] F.C. Alcaraz, M.N. Barber, M.T. Batchelor, R.J. Baxter and G.R.W. Quispel, *Surface exponents of the XXZ, Ashkin-Teller and Potts models*, J. Phys. **A 20** (1987) 6397.

[Ar] H. Araki, *Master symmetries of the XY model*, Commun. Math. Phys.132 (1990) 155-176.

[AS] C. Ahn and K. Shigemoto, *Onsager algebra and integrable lattice models*, Mod. Phys. Lett. **A 6** (1991) 3509.

[Bas1] P. Baseilhac, *Deformed Dolan-Grady relations in quantum integrable models*, Nucl.Phys. **B 709** (2005) 491-521, `arXiv:hep-th/0404149`.

[Bas2] P. Baseilhac, *An integrable structure related with tridiagonal algebras*, Nucl.Phys. **B 705** (2005) 605-619, `arXiv:math-ph/0408025`.

[Bas3] P. Baseilhac, *A family of tridiagonal pairs and related symmetric functions*, J. Phys. **A 39** (2006) 11773.

[BB1] P. Baseilhac and S. Belliard, *Generalized q-Onsager algebras and boundary affine Toda field theories*, Lett. Math. Phys. **93** (2010) 213-228, `arXiv:0906.1215`.

[BB2] P. Baseilhac and S. Belliard, *Central extension of the reflection equations and an analog of Miki's formula*, J. Phys. **A 44** (2011) 415205, `arXiv:1104.1591`.

[BK1] P. Baseilhac and K. Koizumi, *A new (in)finite dimensional algebra for quantum integrable models*, Nucl. Phys. **B 720** (2005) 325-347, `arXiv:math-ph/0503036`.

[BK2] P. Baseilhac and K. Koizumi, *A deformed analogue of Onsager's symmetry in the XXZ open spin chain*, J.Stat.Mech. **0510** (2005) P005, `arXiv:hep-th/0507053`.

[BK3] P. Baseilhac and K. Koizumi, *Exact spectrum of the XXZ open spin chain from the q-Onsager algebra representation theory*, J. Stat. Mech. (2007) P09006, `arXiv:hep-th/0703106`.

[BB3] P. Baseilhac and S. Belliard, *A note on the $O_q(\widehat{sl_2})$ algebra*, `arXiv:1012.5261`.

[BR] S. Belliard and E. Ragoucy, *The nested Bethe ansatz for 'all' open spin chains with diagonal boundary conditions*, J. Phys. **A 42** (2009) 205203; `arXiv:0902.0321`.

[BSh] P. Baseilhac and K. Shigechi, *A new current algebra and the reflection equation*, Lett. Math. Phys. **92** (2010) 47-65, `arXiv:0906.1482`.

[BF] S. Belliard and V. Fomin, *Generalized $q-$Onsager algebras and dynamical K matrices*, J. Phys. A: Math. Theor. **45** (2012) 025201, `arXiv:1106.1317`.

[bXXZ] R.I. Nepomechie, *Bethe Ansatz solution of the open XXZ chain with nondiagonal boundary terms*, J. Phys. **A 37** (2004) 433-440; `arXiv:hep-th/0304092`;
R. Murgan, R.I. Nepomechie and C. Shi, *Exact solution of the open XXZ chain with general integrable boundary terms at roots of unity*, JSTAT **0608** (2006) 006; `arXiv:hep-th/0605223`;
J. Cao, H-Q. Lin, K-J. Shi and Y. Wang, *Exact solutions and elementary excitations in the XXZ spin chain with unparallel boundary fields*, Nucl. Phys. **B 663** (2003) 487; `arXiv:cond-mat/0212163`;
W-L. Yang and Y-Z. Zhang, *On the second reference state and complete eigenstates of the open XXZ chain*, JHEP **04** (2007) 044; `arXiv:hep-th/0703222`.

[BCRS] N. Crampé, E. Ragoucy and D. Simon, *Eigenvectors of open XXZ and ASEP models for a class of non-diagonal boundary conditions*, J. Stat. Mech. (2010) P11038; `arXiv:1009.4119`;
S. Belliard, N. Crampé and E. Ragoucy, Algebraic Bethe ansatz for open XXX model with triangular boundary matrices, `arXiv:1209.4269`.

[Dav] B. Davies, *Onsager's algebra and superintegrability*, J. Phys. **A 23** (1990) 2245-2261;
B. Davies, *Onsager's algebra and the Dolan-Grady condition in the non-self-dual case*, J. Math. Phys. **32** (1991) 2945-2950.

[DFJMN] M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki, *Correlation functions of the XXZ model for $\Delta < -1$* , Phys Lett. A 168 (1992) 256, `arXiv:hep-th/9205055`;
B. Davies, O. Foda, M. Jimbo, T. Miwa and A. Nakayashiki, *Diagonalization of the XXZ Hamiltonian by vertex operators* , Commun. Math. Phys **151** (1993) 89, `arXiv:hep-th/9204064`;
M. Jimbo, T. Miwa, *qKZ equation with $|q| = 1$ and correlation functions of the XXZ model in the gapless regime*, J. Phys. **A 29** (1996) 2923, `arXiv:hep-th/9601135`.

[Do] A. Doikou, *Boundary non-local charges from the open spin chain*, J. Stat. Mech. (2005) P12005, `math-ph/0402067`.

[FM] O. Foda and T. Miwa, *Corner transfer matrices and quantum affine algebras*, Int. J. Mod. Phys. **A 7** Suppl. 1A (1992), 279-302. `arXiv:hep-th/9204068v1`.

[Gal] W. Galleas, *Functional relations from the Yang-Baxter algebra: Eigenvalues of the XXZ model with non-diagonal twisted and open boundary conditions*, `arXiv:0708.0009`.

[IIJMNT] M. Idzumi, K. Iohara, M. Jimbo, T. Miwa, T. Nakashima and T. Tokihiro, *Quantum affine symmetry in vertex models*, Int. Jour. Mod. Phys. **8** (1993) 8, 1479-1511; `arXiv:hep-th/9208066v1`.

[IT] T. Ito and P. Terwilliger, *The augmented tridiagonal algebra*, Kyushu Journal of Mathematics, **64** (2010) No. 1, 81-144, `arXiv:0904.2889v1`.

[Ji] M. Jimbo, *Quantum group symmetry and lattice correlation functions*, Chinese Journal of Physics **30** (1992) 7, 973-985.

[JKKKMW] M. Jimbo, R. Kedem, T. Kojima, H. Konno and T. Miwa, *XXZ chain with a boundary*, Nucl. Phys. B 441 (1995) 437-470;
M. Jimbo, R. Kedem, H. Konno, T. Miwa and R. Weston, *Difference Equations in Spin Chains with a Boundary*, Nucl. Phys. B 448 (1995) 429-456.

[JKKKM] M. Jimbo, R. Kedem, T. Kojima, H. Konno and T. Miwa, *XXZ chain with boundary*, Nucl. Phys. B **441** (1995) 437-470, `arXiv:hep-th/9411112v3`.

[Kau] H.G. Kausch, *Curiosities at* $c = -2$, hep-th/9510149.

[KBPZ] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nucl. Phys. **B 241** (1984) 333;
V.G. Knizhnik and A.B. Zamolodchikov, *Current algebra and Wess-Zumino model in two dimensions*, Nucl. Phys. **B 247** (1984) 83;
Ph. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*, Springer.

[KKMNST] N. Kitanine, K.K. Kozlowski, J.M. Maillet, G. Niccoli, N.A. Slavnov and V. Terras, *Correlation functions of the open XXZ chain I*, JSTAT 0710 (2007) P10009; arXiv:0707.1995;
N. Kitanine, K. Kozlowski, J. M. Maillet, G. Niccoli, N. A. Slavnov and V. Terras, *Correlation functions of the open XXZ chain II*, JSTAT **07** (2008) P07010; arXiv:0803.3305.

[Ko] T. Kojima, *A remark on ground state of boundary Izergin-Korepin model*, arXiv:1101.4078;
T. Kojima, *Free field approach to diagonalization of boundary transfer matrix : recent advances* , arXiv:1103.5526.

[Kol] S. Kolb, *Quantum symmetric Kac-Moody pairs*, arXiv:12067.6036.

[LP] S. Lukyanov and Y. Pugai, *Bosonization of ZF algebras: direction toward deformed Virasoro algebra*, arXiv:hep-th/9412128v1.

[MMR] C.S. Melo, M.J. Martins and G.A.P. Ribeiro, *Bethe ansatz for the XXX-S chain with non-diagonal open boundaries*, Nucl. Phys. **B 711** (2005) 565; arXiv:nlin/0411038.

[M] K. Miki, *Creation/annihilation operators and form factors of the XXZ model*, Physics letters. **A 186** (1994) 186, 217-224.

[MRS] A. Molev, E. Ragoucy and P. Sorba, *Coideal subalgebras in quantum affine algebras*, Rev. Math. Phys **15** (2003), 789822; arXiv:math/0208140.

[N] G. Niccoli, *Antiperiodic spin-1/2 XXZ quantum chains by separation of variables: Complete spectrum and form factors*; arXiv:1205.4537;
Non-diagonal open spin-1/2 XXZ quantum chains by separation of variables: Complete spectrum and matrix elements of some quasi-local operators; arXiv:1206.0646.

[NRGP] A. Nichols, V. Rittenberg and J. de Gier, *One-boundary Temperley-Lieb algebras in the XXZ and loop models*, JSTAT **0503** (2005) 003;
J. de Gier, A. Nichols, P. Pyatov and V. Rittenberg and *Magic in the spectra of XXZ quantum chain at* $\Delta = 0$ *and* $\Delta = -1/2$, hep-th/0505062.

[Ons] L. Onsager, *Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition*, Phys. Rev. **65** (1944) 117-149.

[PeAMT] J.H.H. Perk, "*Star-triangle equations, quantum Lax operators, and higher genus curves*", Proceedings 1987 Summer Research Institute on Theta functions, Proc. Symp. Pure. Math. Vol. 49, part 1. (Am. Math. Soc., Providence, R.I., 1989), 341-354;
H. Au-Yang, B.M. McCoy, J.H.H. Perk and S. Tang, "*Solvable models in statistical mechanics and Riemann surfaces of genus greater than one*" in Algebraic Analysis, Vol. 1, M. Kashiwara and T. Kawai, eds., Academic Press, San Diego, 1988, 29-40.

[PS] V. Pasquier and H. Saleur, *Common structures between finite systems and conformal field theories through quantum groups*, Nucl. Phys. **B 330** (1990) 523-556.

[vGR] G. von Gehlen and V. Rittenberg, Z_n*-symmetric quantum chains with infinite set of conserved charges and* Z_n *zero modes*, Nucl. Phys. **B 257** [FS14] (1985) 351.

[R] V. Regelskis, *Reflection algebras for* $SL(2)$ *and* $GL(1|1)$, arXiv:1206.6498.

[Sk] E.K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. **A 21** (1988) 2375-2389.

[Ter2] P. Terwilliger,*Two relations that generalize the* $q-$*Serre relations and the Dolan-Grady relations*, Proceedings of the Nagoya 1999 International workshop on physics and combinatorics. Editors A. N. Kirillov, A. Tsuchiya, H. Umemura. 377-398, math.QA/0307016.

Laboratoire de Mathématiques et Physique Théorique CNRS/UMR 7350, Fédération Denis Poisson, Université de Tours, Parc de Grammont, 37200 Tours, FRANCE

E-mail address: baseilha@lmpt.univ-tours.fr

Laboratoire Charles Coulomb CNRS/UMR 5221, Université Montpellier 2, F-34095 Montpellier, FRANCE

E-mail address: samuel.belliard@univ-montp2.fr